

Exercise 1. Let R be a ring and let M, K, L and N be R -modules. Assume that $\text{Ext}_R^i(M, N)$, $\text{Ext}_R^i(K, N)$ and $\text{Ext}_R^i(L, N)$ have finite length for all $i \geq 0$, and that there exist integers r, s such that they are all zero for all $i > s$. Show that if

$$0 \longrightarrow K \longrightarrow M \longrightarrow L \longrightarrow 0$$

is a short exact sequence, then

$$\sum_{i=0}^s (-1)^i \text{length } \text{Ext}_R^i(M, N) = \sum_{i=0}^s (-1)^i \text{length } \text{Ext}_R^i(K, N) + \sum_{i=0}^s (-1)^i \text{length } \text{Ext}_R^i(L, N).$$

Exercise 2 (Nullstellensatz for $\text{Spec } R$). Let R be a commutative ring. Given a closed subset $Z \subseteq \text{Spec } R$, define $I(Z) := \{f \in R, Z \subseteq V(f)\}$. Show that $I(Z)$ is an ideal, and that for all ideals $I \subseteq \text{Spec } R$,

$$I(V(I)) = \sqrt{I}$$

In particular, show that for all ideals I, J of R ,

$$V(I) = V(J) \iff \sqrt{I} = \sqrt{J}$$

Exercise 3. Let R be a commutative ring and $I \subseteq R$ be a radical ideal. Show that I is prime if and only if $V(I)$ is an irreducible topological space.

Exercise 4. Let $R = \mathbb{C}[x, y, z]$ and $I = (xy - z^2, x^2 - y^2) \subseteq R$. Identify $V(I) \subset \mathbb{C}^3$. Notice that this naturally breaks into smaller algebraic sets. What are the ideals of each piece?

Exercise 5. Let F be an algebraically closed field. Let X and Y be algebraic sets in F^n .

- (1) Prove that $I(X \cup Y) = I(X) \cap I(Y)$
- (2) By considering $X = V(x^2 - y)$ and $Y = V(y)$ for the ideals $(x^2 - y)$ and (y) in $F[x, y]$, show that it need not be true that $I(X \cap Y) = I(X) + I(Y)$.
- (3) Prove that in general $\sqrt{I(X) + I(Y)} = I(X \cap Y)$.

Review exercises for material from “Anneaux et corps”

Exercise 6. Show that $x^3 + y^7 \in k[x, y]$ is irreducible.

[Hint: Use the consequence of Gauss’s theorem saying that for a unique factorisation domain R and a primitive polynomial $f \in R[t]$, we have that f is irreducible in $\text{Frac}(R)[t]$ if and only if it is irreducible in $R[t]$.]

Exercise 7. Let $R = k[x, y, z]$. Show that $(xz^3 + yz^3 - y^2z^2 + xyz - xy)$ is a prime ideal of R .

[Hint: Use Eisenstein’s Criterion.]

Exercise 8. Solve the following exercises:

- (1) Consider the polynomial $f = X^3Y + X^2Y^2 + Y^3 - Y^2 - X - Y + 1$ in $\mathbb{C}[X, Y]$. Write it as an element of $(\mathbb{C}[X])[Y]$, that is collect together terms according

to powers of Y , and then use Eisenstein's criterion to show that f is prime in $\mathbb{C}[X, Y]$.

- (2) Let F be any field. Show that the polynomial $f = X^2 + Y^2 - 1$ is irreducible in $F[X, Y]$, unless F has characteristic 2. What happens in that case?

Exercise 9. Show the following:

- (1) Let $F \subseteq L$ be a field extension, and suppose $a_1, \dots, a_n$ are elements of L which are algebraically independent over F . Prove that $F(a_1, \dots, a_n)$ is isomorphic to the fraction field of the polynomial ring $F[x_1, \dots, x_n]$.
- (2) Let $F \subseteq L$ be a field extension. Show that a subset of L is a transcendence basis for L over F if and only if it is a maximal algebraically independent set. As a consequence, use Zorn lemma to show that a transcendence basis exists for any field extension $F \subseteq L$.

Exercise 10. Prove that if $F \subseteq K \subseteq L$ are field extensions such that $\text{trdeg}_F L < \infty$, then $\text{trdeg}_F L = \text{trdeg}_F K + \text{trdeg}_K L$