

**Exercise 1.** Let  $F$  be an algebraically closed field, and let  $I, J$  be ideals of  $R = F[x_1, \dots, x_n]$ . Prove that  $\sqrt{I} \subseteq \sqrt{J}$  if and only if  $V(J) \subseteq V(I)$ .

**Exercise 2.** Let  $F$  be an algebraically closed field, and let  $I, J$  be ideals of  $R = F[x_1, \dots, x_n]$ . Show that

- (1)  $V(I) \cup V(J) = V(I \cap J) = V(IJ)$
- (2)  $V(I) \cap V(J) = V(I + J)$

**Exercise 3.** Let  $R$  be a commutative ring, and let  $I, J$  be ideals of  $R$ . In both  $\text{Spec}(R)$  and  $m - \text{Spec}(R)$ , show that

- (1)  $V(I) \cup V(J) = V(I \cap J) = V(IJ)$
- (2)  $V(I) \cap V(J) = V(I + J)$

**Exercise 4.**    ◦ Let  $R, S$  be commutative rings, and let  $f : R \rightarrow S$  be a ring morphism. Show that there is an induced continuous map  $\text{Spec}(S) \rightarrow \text{Spec}(R)$ .  
 ◦ Let  $R$  be a ring and  $I$  an ideal. Show that the morphism  $\text{Spec}(R/I) \rightarrow \text{Spec}(R)$  induced by the quotient map corresponds to the inclusion of the closed subset  $V(I) \subseteq \text{Spec}(R)$ .

**Exercise 5.** Prove that  $Z = \{(u^3, u^2v, uv^2, v^3) : u, v \in \mathbb{C}\} \subset \mathbb{C}^4$  is an algebraic set (i.e. there exists an ideal  $I$  of  $\mathbb{C}[x_1, x_2, x_3, x_4]$  such that  $Z = V(I)$ ). Find  $I(Z)$ .  
 [Hint: Make sure you have everything!]

**Exercise 6.** Let  $F$  be an algebraically closed field, and  $X \subseteq F^m$  an algebraic set with ideal  $I = I(X)$ . Define the coordinate ring  $A(X)$  of  $X$  to be  $A(X) := F[x_1, \dots, x_m]/I$ . Notice that every element of  $A(X)$  naturally defines a set-map from  $X$  to  $F$ , and thus one may think of  $A(X)$  as the set of global algebraic functions on  $X$ .

- (1) If  $X = V(I) \subseteq F^m$ , and  $Y = V(J) \subseteq F^n$  are algebraic sets with ideals  $I = I(X)$  and  $J = I(Y)$ , then a morphism  $f : X \rightarrow Y$  is defined to be a set-map from the points of  $X$  to the points of  $Y$ , for which the following holds: there exists a vector  $(h_1, \dots, h_n)$  of polynomials  $h_i \in F[x_1, \dots, x_m]$ , such that for every  $\underline{a} \in X$  we have  $f(\underline{a}) = (h_1(\underline{a}), h_2(\underline{a}), \dots, h_n(\underline{a})) \in Y$ .

Show that whenever there is a morphism  $f : X \rightarrow Y$  of algebraic sets as defined above, there is a unique homomorphism of  $F$ -algebras  $\lambda_f : A(Y) \rightarrow A(X)$ , such that the following diagram commutes.

$$\begin{array}{ccc}
 F[y_1, \dots, y_n] & \xrightarrow{y_i \mapsto h_i} & F[x_1, \dots, x_m] \\
 \downarrow & & \downarrow \\
 A(Y) & \xrightarrow{\lambda_f} & A(X)
 \end{array}$$

Here the vertical arrows are the quotient maps stemming from the definition of  $A(X)$  and  $A(Y)$ , and the top horizontal map is given by sending  $y_i$  to  $h_i(x_1, \dots, x_m)$ .

- (2) With setup as above, show that if there is a homomorphism of  $F$ -algebras  $\lambda : A(Y) \rightarrow A(X)$ , then there is a morphism  $f : X \rightarrow Y$  such that  $\lambda = \lambda_f$ . Furthermore, all choices of  $f$  are the same (as set-maps from the points of  $X$  to the points of  $Y$ ).

**Exercise 7.** Let  $F$  be an algebraically closed field. Let  $X$  be an algebraic set in  $F^n$  with ideal  $I(X) = I$ . Prove that points of  $F^n$  contained in  $X$  are naturally in bijection with maximal ideals of the coordinate ring  $A(X) = F[x_1, \dots, x_n]/I$ .