EPFL - Fall 2024	Domenico Valloni
Rings and modules	**Exercises**
Sheet 6	31 October 2024

The exercise marked by ♠ is this week's bonus exercise. You can hand in your LaTeX-solutions on Moodle until Wednesday the 13th of November at 6pm sharp.

Throughout, we only work with commutative rings.

Exercise 1. Let $R = k[x, y]$ be the polynomial ring in two variables over an algebraically closed field k. Recall that an ideal $\mathfrak{m}$ in a ring R is maximal if it is not properly contained in any other proper ideal of R. In this exercise you can use freely the Theorem below, which will be proven later in the course.

Theorem (The weak Nullstellensatz in two variables)**.** *Let k be an algebraically closed field. Every maximal ideal $\mathfrak{m}$ in the ring $k[x, y]$ is of the form $\mathfrak{m} = (x-a, y-b)$ for some $a, b \in k$.*

Show the following:

(1) If M is a finite length module over R, then the quotients of its composition series are of the form $R/(x-a, y-b)$.
(2) If M is a module such that $\mathrm{Ann}(M) \supseteq (x-a, y-b)$, then $\mathrm{Ann}(\mathrm{Ext}^i(M, N)) \supseteq (x-a, y-b)$ for every R-module N.
[*Hint*: Consider the multiplication by $x-a$ resp. $y-b$ on M and the induced maps on $\mathrm{Ext}^i_R(M, N)$. Recall also Exercise 7 of Sheet 4.]
(3) If N is any finitely generated module over R, then $\mathrm{Ext}^i\left(R/(x-a, y-b), N\right)$ has finite length.
[*Hint*: Use the previous point.]
(4) For every finite length module M and for every finitely generated module N over R, $\mathrm{Ext}^i_R(M, N)$ has finite length.
[*Hint*: Use the long exact sequence for a compostion series.]

Exercise 2. Let $R = k[x, y]$ be as in the previous exercise (k is algebaically closed). We say that a finite length module is supported at $(x-a, y-b)$ if only $R/(x-a, y-b)$ appears as quotients in the composition series. Show that if M is a finite length module supported at $(x-a, y-b)$, then $\mathrm{Ext}^i_R\left(M, R/(x-a', y-b')\right) = 0$ for all $(a', b') \neq (a, b)$.

Exercise 3. Show using the long exact sequence of cohomology that if $\mathrm{Ext}^1_R(M, N) = 0$, then every extension $0 \longrightarrow N \longrightarrow K \longrightarrow M \longrightarrow 0$ splits.

Exercise 4. ♠ Let $R = k[x, y]$, and let $M = R/(x, y)$.

(1) Show that $\mathrm{Ext}^1_R(M, M) \cong M^2$.

Note that there is canonical bijection $k \to M$, sending $\lambda \in k$ to the class of the constant polynomial λ modulo (x, y). In particular, there is also a natural bijection $k^2 \to M^2$.

(2) For a given $(\lambda, \mu) \in k^2 \setminus \{(0, 0)\}$, define
$$N_{\lambda,\mu} = R/(x^2, y^2, xy, \lambda y - \mu x),$$
let $\varphi\colon N_{\lambda,\mu} \to M$ be the map induced by the quotient map $R \to M$, and let $\psi\colon M \to N_{\lambda,\mu}$ be the map sending the class of 1 to the class of $-(xa + yb)$, where $a, b \in k$ are any elements such that $\lambda a + \mu b = 1$.

Then show that the Yoneda extension associated to $(\lambda, \mu) \in k^2 \setminus \{(0,0)\}$ is isomorphic to the sequence

$$0 \to M \xrightarrow{\psi} N_{\lambda,\mu} \xrightarrow{\varphi} M \to 0.$$

(3) Under what conditions on (λ, μ) and (λ', μ') do we have an isomorphism $N_{\lambda,\mu} \cong N_{\lambda',\mu'}$?
Hint: Think about torsion.

Exercise 5. Let $R = k[x, y]$.

(1) Show that $\mathrm{Ext}^1\left((x,y), {}^{R}\!/_{(x,y)}\right) \neq 0$.
(2) Construct a finitely generated module M such that $\mathrm{Tors}(M) \subseteq M$ is not a direct summand.

[*Note:* For M finitely generated over a PID R, $\mathrm{Tors}(M) \subseteq M$ is always a direct summand by the fundamental theorem for finitely generated modules over PIDs.]

Exercise 6. Throughout this exercise, R will be a ring and M, N will be R-modules. We will now see another way to compute the Ext-modules than the one we saw in the lectures (one may say a 'dual' way). To do so, we need the following Lemma, which you may use without proof.

Lemma 1. [1] *For every R-module N there exists an injective R-module homomorphism $N \to I$ where I is an injective R-module.*

(1) Using the above Lemma, show that any R-module N admits an injective resolution. That is, there exists an exact sequence

$$0 \longrightarrow N \xrightarrow{i^{-1}} I^0 \xrightarrow{i^0} I^1 \longrightarrow \cdots$$

where I^b is an injective R-module for all $b \geq 0$ (the numbers in superscript are just indices, *not* exponents of any sort).
(2) Show that an R-module I is injective if and only if $\mathrm{Hom}_R(-, I)$ is exact.
[*Reminder:* By Lemma 5.2.2 of the lecture notes $\mathrm{Hom}_R(-, I)$ is always left exact.]

[1] This is not too hard to prove but it needs some preparation. It boils down to proving the result for $R = \mathbb{Z}$ using Bear's criterion, and then generalizing it to any ring by some trickery.

(3) Fix a projective resolution $P_\bullet \twoheadrightarrow M$ and an injective resolution $N \hookrightarrow I^\bullet$. Consider the commutative diagram

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \operatorname{Hom}_R(M, I^1) \arrow[u] \arrow[r, "d_{-1,1}"] & \operatorname{Hom}_R(P_0, I^1) \arrow[u] \arrow[r, "d_{0,1}"] & \operatorname{Hom}_R(P_1, I^1) \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & \operatorname{Hom}_R(M, I^0) \arrow[u, "\delta_{-1,0}"'] \arrow[r, "d_{-1,0}"] & \operatorname{Hom}_R(P_0, I^0) \arrow[u, "\delta_{0,0}"'] \arrow[r, "d_{0,0}"] & \operatorname{Hom}_R(P_1, I^0) \arrow[u, "\delta_{1,0}"'] \arrow[r] & \cdots \\
 & 0 \arrow[u] \arrow[r] & \operatorname{Hom}_R(P_0, N) \arrow[u, "\delta_{0,-1}"'] \arrow[r, "d_{0,-1}"] & \operatorname{Hom}_R(P_1, N) \arrow[u, "\delta_{1,-1}"'] \arrow[r] & \cdots \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where $d_{a,b} = - \circ p_{a+1}$ and $\delta_{a,b} = i^b \circ -$ for all $a, b \geq -1$. Briefly justify that this is indeed commutative, and that all columns and lines of the diagram which are not blue are exact.

(4) Show that $H^0(\operatorname{Hom}_R(M, I^\bullet)) \cong H^0(\operatorname{Hom}_R(P_\bullet, N))$.
[*Hint:* Show that their images inside $\operatorname{Hom}_R(P_0, I^0)$ coincide.]

(5) Show that $H^1(\operatorname{Hom}_R(M, I^\bullet)) \cong H^1(\operatorname{Hom}_R(P_\bullet, N))$.
[*Hint:* Let $C^0 := \operatorname{Hom}_R(P_0, I^0)$ and $C^1 = \operatorname{Hom}_R(P_1, I^0) \oplus \operatorname{Hom}_R(P_0, I^1)$, and let $\Delta^0 : C^0 \to C^1$ be the map sending $x \in C^0$ to $(d_{0,0}(x), \delta_{0,0}(x)) \in C^1$. Show that the cohomology groups in question both embed into $\operatorname{coker}(\Delta^0)$ and that their images therein coincide.]

[*Remark:* One can generalize the above results and prove that in fact $H^i(\operatorname{Hom}_R(M, I^\bullet)) \cong H^i(\operatorname{Hom}_R(P_\bullet, N))$ for all $i \geq 0$, and thus the Ext-modules may also be computed by using an injective resolution of the second module. To do so, one defines the modules $C^m := \bigoplus_{a+b=m} \operatorname{Hom}_R(P_a, I^b)$ and connecting maps $\Delta^m : C^m \to C^{m+1}$ similar to Δ^0, where one replaces $\delta_{a,b}$ by $(-1)^a \delta_{a,b}$ to ensure $\Delta^{m+1} \circ \Delta^m = 0$. We thus obtain a complex $C^\bullet$, and one can then prove that $H^i(\operatorname{Hom}_R(M, I^\bullet))$ and $H^i(\operatorname{Hom}_R(P_\bullet, N))$ embed into $H^i(C^\bullet)$ with equal image.]