EPFL - Fall 2024	Domenico Valloni
Rings and modules	**Exercises**
Sheet 5	17 October 2024

Exercise 1. For two short exact sequences

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

and

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

we say that there is a map between them if there exists morphisms $f_i : M_i \to N_i$, for $1 \le i \le 3$ and a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, "f_1"] & M_2 \arrow[r] \arrow[d, "f_2"] & M_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & 0.
\end{tikzcd}$$

Show that whenever there is a map between two short exact sequences, then there is an induced map between long exact sequences of Ext-modules, making the suitable diagram commute.

Exercise 2. In this exercise we prove the the two *4-lemmas*. To this end, suppose that we have a commuting diagram with exact rows:

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "a"] & B \arrow[r, "f_2"] \arrow[d, "b"] & C \arrow[r, "f_3"] \arrow[d, "c"] & D \arrow[d, "d"] \\
A' \arrow[r, "f_1'"] & B' \arrow[r, "f_2'"] & C' \arrow[r, "f_3'"] & D'
\end{tikzcd}$$

(1) Show that if a and c are surjective and d is injective, then b is an surjective.
(2) Show that if b and d are injective and a is surjective, then c is a injective.

Exercise 3. (1) Set $k = \mathbb{F}_p$ and $G = \mathbb{Z}/p\mathbb{Z}$. Find all the submodules (i.e. ideals) of $R = k[G]$. [1]
[*Hint:* To understand $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$ in terms of more common rings, it might be a good idea to look for ring morphisms $\mathbb{F}_p[x] \to \mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$ and investigate both kernel and image.]
(2) For $p = 2$, let x denote a generator of G and set $M = (x+1) \subseteq k[G]$. Compute $\mathrm{Ext}^i_R(M, M)$ for all $i \ge 0$.

[1] Recall that $k[G]$ is defined as the set of functions $f : G \to k$, where addition is defined pointwise, and multiplication is defined by convolution:

$$(f \cdot g)(z) := \sum_{\substack{x,y \in G \\ xy = z}} f(x)g(y) \quad \text{for all } f, g \in k[G],\ z \in G.$$

Now for $g \in G$, let $\delta_g \in k[G]$ be the function taking the value 1 at g and 0 everywhere else. Then $\{\delta_g\}_{g \in G}$ is a k-basis of $k[G]$. Furthermore, one can verify that the map $g \mapsto \delta_g$ is an injective group morphism $G \hookrightarrow k[G]^\times$ (in particular, $1_{k[G]} = \delta_{e_G}$). Usually people just write g instead of δ_g by abuse of notation, but as the underlying sets of $\mathbb{F}_p$ and $\mathbb{Z}/p\mathbb{Z}$ coincide, it is probably better to distinguish them in the notation.

Exercise 4. In this exercise we define injective modules and prove *Baer's criterion.* Let R be a (not necessarily commutative) ring; any R-module and any R-morphism appearing in this exercise will be a left R-module resp. a morphism of left R-modules.
We say that an R-module Q is injective if it satisfies the following property:
Whenever we have an injective R-morphism $f : X \hookrightarrow Y$ and an R-morphism $g : X \to Q$, then there exists an R-morphism $h : Y \to Q$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, hook, "f"] \arrow[d, "g"] & Y \arrow[dl, dotted, "h"] \\ Q & \end{tikzcd}$$

We will prove the following:

Theorem (Baer's Criterion)**.** *Suppose that the left R-module Q has the property that if I is any left ideal of R and $f : I \to Q$ is an R-morphism, there exists an R-morphism $F : R \to Q$ extending f. Then Q is an injective R-module.*

We will prove *Baer's criterion* in several steps. Assume that the R-module Q satisfies Baer's criterion.

(1) Let X, Y be R-modules, and assume that Y is *cyclic* (generated by $b \in Y$). Let $f : X \hookrightarrow Y$ be an injective R-morphism. Show that for every R-morphism $g : X \to Q$, there exists an R-morphism $h : Y \to Q$ making the appropriate diagram commute.
[*Hint:* Identify X with a submodule of Y and consider the subset I of R defined by $I = \{r \in R : rb \in X\}$.]
(2) Let X, Y be left R-modules with an injective R-morphism $f : X \hookrightarrow Y$ (we identify X with its image under f). Let $b \in Y$ be arbitrary. With a similar approach as in the previous point, prove that any R-morphism $g : X \to Q$ can be extended to an R-morphism $h : X + Rb \to Q$ making the appropriate diagram commute.
(3) Use *Zorn's Lemma* to conclude the proof.

Axiom 1 (Zorn's Lemma / Axiom of Choice)**.** If $(\mathcal{P}, \leq)$ is a partially ordered set with the property that every totally ordered subset (often called a chain) has an upper bound, then there exists a maximal $M \in \mathcal{P}$. (that is, for $N \in \mathcal{P}$, we have $M \not\leq N$)

[*Hint:* Try to think of what it means for one partial extension of $g : X \to Q$ to be smaller than another.]

Exercise 5. Use Baer's Criterion to show that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.