EPFL - Fall 2024	Domenico Valloni
Rings and modules	**Exercises**
Sheet 4	10 October 2024

The exercise marked by ♠ is this weeks bonus exercise. You can hand in your LaTeX-solutions on Moodle until Wednesday the 30th of October at 6pm sharp.

Exercise 1. Let R be a commutative ring, and let M be an R-module.

(1) Show that $\mathrm{Hom}_R(M, -)$ is *left exact.* That is, for any short exact sequence of R-modules

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0\ ,$$

there is an induced exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, N') \longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(M, N'')\ .$$

(2) Give an example of a ring R and an R-module M such that $\mathrm{Hom}_R(M, -)$ is not *right exact.* That is, give an example of a surjection of R-modules $N \twoheadrightarrow N''$ such that the induced morphism $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N'')$ is not surjective.

Exercise 2. Let $R = k[x, y]$ where k is a field. Extend the complex below to a free resolution $F_\bullet$ of the R-module $k \cong R/(x, y)$. Then compute $\mathrm{Ext}^i_{F_\bullet}(k, R)$ for each i, and note that you get the same as for the resolutions in Example 5.3.9 in the printed course notes.

$$R \oplus R \oplus R \longrightarrow R \longrightarrow k \longrightarrow 0$$

The first morphism is defined by sending a basis to the following elements:

$$(1, 0, 0) \mapsto x, (0, 1, 0) \mapsto y, (0, 0, 1) \mapsto x + y$$

and the second morphism is the natural surjection $R \to k$.
[*Remark:* This is an example of the fact that the Ext-modules $\mathrm{Ext}^i_{F_\bullet}(M, N)$ don't depend on the free resolution $F_\bullet$ of M.]

Exercise 3. Let $0 \to M \xrightarrow{i} Z \xrightarrow{p} N \to 0$ be a short exact sequence of R-modules.

(1) A *section* of p is a morphism $s\colon N \to Z$ such that $p \circ s = \mathrm{id}_N$. Show that p admits a section if and only if there exists an isomorphism $\Phi : M \oplus N \xrightarrow{\cong} Z$ and a commuting diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, equal] & Z \arrow[r, "p"] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "e"] & M \oplus N \arrow[r, "\pi"] \arrow[u, "\Phi"] & N \arrow[r] & 0
\end{tikzcd}$$

(2) A *section* of i is a morphism $q\colon Z \to M$ such that $q \circ i = \mathrm{id}_M$. Show that i admits a section if and only if there exists an isomorphism $\Psi : Z \xrightarrow{\cong} M \oplus N$ and a commuting diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, equal] & Z \arrow[r, "p"] \arrow[d, "\Psi"] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "e"] & M \oplus N \arrow[r, "\pi"] & N \arrow[r] & 0
\end{tikzcd}$$

We say that a short exact sequence satisfying any of these conditions is *split exact*.

Exercise 4. Consider the ring $\mathbb{Z}[\sqrt{-5}]$.

(1) Is the ideal $(2, 1+\sqrt{-5})$ a free $\mathbb{Z}[\sqrt{-5}]$-module?
[*Hint:* Consider the element $6 \in \mathbb{Z}[\sqrt{-5}]$.]
(2) Prove that $(2, 1+\sqrt{-5})$ is a projective $\mathbb{Z}[\sqrt{-5}]$-module.
[*Hint*: Prove that $(2, 1+\sqrt{-5})$ is projective by showing that it is a direct summand of a free module. To do this, define the obvious surjection $p : \mathbb{Z}[\sqrt{-5}]^2 \to (2, 1+\sqrt{-5})$ and examine the assignment $s : (2, 1+\sqrt{-5}) \to \mathbb{Z}[\sqrt{-5}]^2$ defined by $s(x) = 2xe_1 - \frac{1-\sqrt{-5}}{2}xe_2$.]

Exercise 5. ♠ Let R be a commutative ring. The *projective dimension* of an R-module M is the smallest integer $n \geq 0$ such that there exists a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0$$

of M. We write $\operatorname{projdim}(M) = n$, and if no finite projective resolution exists, this number is by definition ∞.

In our case, we focus on the ring $R = k[x,y]/(x^2 - y^3)$ and $M = R/(x,y)$. The goal is to show that M does not have finite projective dimension. Proceed as follows:

(1) Compute the dimension as a k-vector space of $\operatorname{Ext}^1_R(M,M)$.
(2) Show that there is a short exact sequence

$$0 \to M \to R/y \to M \to 0.$$

(3) Use the two points above to show that $\operatorname{Ext}^i_R(M,M) \neq 0$ for all $i \geq 0$.
(4) Conclude that $\operatorname{projdim}(M) = \infty$ has no finite projective resolution.

Remark 0.1. A celebrated theorem of Serre states that a ring R is *regular* if and only if every module M over R has finite projective dimension. Without going into details, regular means that the associated algebraic variety looks "good" (e.g. would be a smooth manifold over the complex numbers). This gives a very important application of Ext-functors in commutative algebra, since they help detect the projective dimension of modules (and hence regularity of the ring).

In the case above, note that the associated variety (here $\{(x,y) \in \mathbb{R}^2 | x^2 = y^3\}$ if $k = \mathbb{R}$) doesn't look good at the origin (draw this curve!), it has a so-called cusp singularity, and hence it is not regular. This exercise is then about verifying Serre's theorem in a special example.

Exercise 6. Prove the following.

(1) If $0 \longrightarrow M_n \longrightarrow \ldots \longrightarrow M_0 \longrightarrow 0$ is an exact sequence of finitely generated modules over an Artinian and Noetherian ring R, then $0 = \sum_{i=0}^{n} (-1)^i \operatorname{length} M_i$.
(2) Let $R = k[\varepsilon]$ denote (as usual) the quotient $k[x]/(x^2)$ where k is a field (and ε is the class of x). Let M be the R-module $R/(\varepsilon)$. Show that M has no finite resolution by finitely generated free modules.
(3) In general if R is Artinian and Noetherian, and $\operatorname{length} R \nmid \operatorname{length} M$, prove that M has no finite resolution by finitely generated free modules.
(4) Prove that over a PID every finitely generated module has a finite free resolution.

Exercise 7. In this exercise R is an integral domain which is not a field; in particular it is commutative. Recall the definition of an R-module M being divisible: for all $m \in M$ and $r \in R \setminus \{0\}$ there exists an $n \in M$ such that $rn = m$. In other words, M is divisible if and only if multiplication by r on M is surjective for every $r \in R \setminus \{0\}$.

(1) Show that a non-trivial free R-module is not divisible.

(2) Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module, or in general $\mathrm{Frac}(R)$ is not a projective R-module.
[*Hint:* Define the notion of submodule of divisible elements, and refine (1) by showing that it is trivial for free R-modules.]

(3) From now on, let M, N be R-modules. Let $P_\bullet$ be a projective resolution of M and let $\psi : N \to N$ be the R-module homomorphism corresponding to multiplication by a fixed $r \in R$. Show that ψ induces a co-chain morphism $\mathrm{Hom}_R(P_\bullet, N) \to \mathrm{Hom}_R(P_\bullet, N)$. By passing to cohomology, one obtains a map $\mathrm{Ext}^i_R(M, \psi) : \mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^i_R(M, N)$. Show that $\mathrm{Ext}^i_R(M, \psi)$ is still just multiplication by r on $\mathrm{Ext}^i_R(M, N)$. In particular, it is independent of the projective resolution.
[*Remark:* One can in fact perform an analogous construction for *any* R-module homomorphism $\psi : N \to L$, and thus obtain a map $\mathrm{Ext}^i_R(M, \psi) : \mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^i_R(M, L)$, which as in Remark 5.4.26 of the printed course notes is independent of the projective resolution. This makes also $\mathrm{Ext}^i_R(M, -)$ a functor, while in the course we only saw that $\mathrm{Ext}^i_R(-, N)$ is a functor.]

(4) Fix $r \in R$, and let $\phi : M \to M$ be the multiplication by r. Show that $\mathrm{Ext}^i_R(\phi, N)$, as in Definition 5.4.25 of the course notes, is also just the multiplication by r on $\mathrm{Ext}^i_R(M, N)$.

(5) Show that, despite $\mathrm{Frac}(R)$ being not a projective R-module, if N is an R-module such that $\mathrm{Ann}(N) \neq 0$, then $\mathrm{Ext}^i_R(\mathrm{Frac}(R), N) = 0$ for all $i \geq 0$ (note that for P projective, $\mathrm{Ext}^i_R(P, N) = 0$ for all $i > 0$ by definition).