

Exercise 1. Computing a presentation of an R -module M means explicitly determining an exact sequence of the form $R^{\oplus t} \xrightarrow{\eta} R^{\oplus s} \xrightarrow{\varepsilon} M \longrightarrow 0$. Do the following computations.

- (1) Compute a presentation of the $\mathbb{Z}$ -module

$$M := \mathbb{Z}(2, 9) + \mathbb{Z}(4, 3) + \mathbb{Z}(6, 8) \subseteq \mathbb{Z} \oplus \mathbb{Z}.$$

- (2) Let $R = \text{Mat}_{2 \times 2}(\mathbb{Z})$ be the ring of 2×2 -matrices over $\mathbb{Z}$. Compute a presentation of the left R -module

$$M := R \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + R \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \subseteq R.$$

Exercise 2. Do the following:

- (1) Calculate the Smith normal form of the following matrix over $\mathbb{Z}$.

$$\begin{pmatrix} 1 & 9 & 1 \\ -2 & -6 & 0 \\ 2 & -8 & 2 \\ -1 & 1 & 5 \end{pmatrix}$$

- (2) (i) Find a direct sum of cyclic $\mathbb{Z}$ -modules isomorphic to the $\mathbb{Z}$ -module M with generators e_1, e_2, e_3, e_4 and relations

$$\begin{aligned} e_1 - 2e_2 + 2e_3 - e_4 &= 0 \\ 9e_1 - 6e_2 - 8e_3 + e_4 &= 0 \\ e_1 + 2e_3 + 5e_4 &= 0 \end{aligned}$$

[*Hint/Remark:* By definition, M is the quotient of the free $\mathbb{Z}$ -module on 4 generators $\bigoplus_{i=1}^4 \mathbb{Z}e_i$ by the submodule generated by $e_1 - 2e_2 + 2e_3 - e_4$, $9e_1 - 6e_2 - 8e_3 + e_4$ and $e_1 + 2e_3 + 5e_4$. Notice that in the quotient, $e_1, \dots, e_4$ then satisfy exactly these relations.]

- (ii) Explicitly give 'nice' generators of M , in terms of the original generators e_1, e_2, e_3, e_4 . Here, $f_1, \dots, f_s$ are 'nice' generators if the relations they satisfy are generated by relations of the form $m_i f_i = 0$, where $m_1, \dots, m_s \in \mathbb{Z}$ are integers.

Exercise 3. Let $R = \mathbb{Q}[x]$. Find a direct sum of cyclic R -modules isomorphic to the R -module with generators e_1, e_2 and relations

$$\begin{aligned} x^2 e_1 + (x + 1)e_2 &= 0 \\ (x^3 + 2x + 1)e_1 + (x^2 - 1)e_2 &= 0 \end{aligned}$$

Exercise 4. Give an example of an infinitely generated $\mathbb{Z}$ -module which is *not* an (infinite) direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ for various choices of n .

Remark 0.1. This shows that the fundamental theorem of finitely generated modules over PID's does not hold for infinitely generated modules.

Exercise 5. Let $R = \mathbb{Z}[x]$ and consider the matrix $A = \begin{pmatrix} 2 & x \\ 0 & 0 \end{pmatrix} \in \text{Mat}_{2 \times 2}(R)$.

- (1) Show that A is not equivalent to a diagonal matrix. The equivalence that we consider here is the one introduced in the lectures, that is, up to left or right multiplication by an invertible matrix.
- (2) Show that the cokernel of the map $A : R^{\oplus 2} \rightarrow R^{\oplus 2}$ is isomorphic to a direct sum of cyclic R -modules, but is not isomorphic to an R -module of the form $R^{\oplus m} \oplus \bigoplus_{i=1}^n R/(a_i)$ where $a_1, \dots, a_n \in R \setminus \{0\}$.
- (3) Show that $(2, x)$ is not isomorphic to a direct sum of cyclic R -modules.

Exercise 6. Show that an exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$$

of R -modules induces an exact sequence

$$0 \longrightarrow \text{Tors}(M) \longrightarrow \text{Tors}(N) \longrightarrow \text{Tors}(L) ,$$

but not necessarily an exact sequence

$$0 \longrightarrow \text{Tors}(M) \longrightarrow \text{Tors}(N) \longrightarrow \text{Tors}(L) \longrightarrow 0 .$$

Exercise 7. Let $M \in \text{Mat}(n \times n, k)$ for a field k . Show that there is a basis with respect to which M is block diagonal with blocks of the form

$$\begin{pmatrix} 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & \ddots & 0 & a_1 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & a_{d-2} \\ 0 & 0 & \dots & 1 & a_{d-1} \end{pmatrix}$$

Hint: M acts naturally on some n -dimensional k -vector space V . Consider V as a $k[x]$ -module via $f \cdot v = f(M)(v)$ and use the classification of finitely generated modules over a PID.