EPFL - Fall 2024	Domenico Valloni
Rings and modules	**Exercises**
Sheet 2	26 October 2024

The exercise marked by ♠ is this week's bonus exercise. You can hand in your LaTeX-solutions on Moodle until Wednesday, the 9th of October at 6pm sharp.

Exercise 1. Show that the following holds for a R-module M of finite length $l(M)$ (i.e., an R-module M that admits a composition series of finite length).

(1) If there is a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of R-modules, then $l(M) = l(M') + l(M'')$.

(2) If $N <_R M$ is a proper submodule then $l(N) < l(M)$.

(3) Use (2) to show that any strict chain of submodules in M (not necessary a maximal chain, i.e. not necessarily a composition series) has length smaller than or equal to $l(M)$. Conclude that a module M is of finite length if and only if M is both Noetherian and Artinian.

Exercise 2. Let R be a commutative ring and let M be a finitely generated module over R. Let $f : M \to M$ be an R-module homomorphism.

(1) Suppose that R is a Noetherian ring.
 (i) Does injectivity of f implies surjectivity?
 (ii) Does surjectivity of f implies injectivity?

(2) Suppose that M is a module of finite length, show that f is injective if and only if f is surjective.

Exercise 3. (1) Let R be a PID, and let $f \in R$ be a product of $n \geq 0$ prime elements. Prove that the length of $R/(f)$ as an R-module is equal to n.

(2) Let $f \in \mathbb{R}[x]$ be a nonzero polynomial with exactly $n \geq 0$ non-real roots (counted with multiplicity). Prove that

$$\dim_{\mathbb{R}}\left(\mathbb{R}[x]/(f)\right) - \mathrm{length}_{\mathbb{R}[x]}\left(\mathbb{R}[x]/(f)\right) = n/2$$

(3) Let M be a $\mathbb{Z}$-module. Prove that M has finite length if and only if it is finite (as a set).

(4) Give an example of a ring and a module over this ring which has finite length but infinitely many submodules.

Exercise 4. (1) Let $n, m > 0$ be integers, let k be a field and let $R := k[x, y]$. Show that the R-module

$$M := k[x, y]/(x^n, y^m)$$

has length nm.

Hint: Exercise 1 can be useful to decompose this computation into easier ones, allowing some induction argument. The same applies for the next point.

(2) Let $p > 0$ be a prime number. Compute the length of

$$\mathbb{Z}[x]/(p^2, x^2 - p),$$

as a module over the ring $\mathbb{Z}[x]$.

Exercise 5. ♠ Compute the length of the $\mathbb{C}[x,y,z]$-module module
$$M := \mathbb{C}[x,y,z]\Big/\left(x^3+3x^2+2xy, y^2-1, z^{2024}\right).$$

Exercise 6. Let R be a commutative Noetherian ring. Are the following rings Noetherian? Are they Artinian?

(1) $R[x, \frac{1}{x}] := \{\sum_{i=-m}^{n} a_i x^i : a_i \in R, m, n \in \mathbb{N}\}$.
(2) $R[x_1, x_2, x_3, \ldots]$.
(3) $R[[x]]$, the ring of formal power series[1] with coefficients in R.
Hint: For an ideal I and each $n \in \mathbb{N}$, let $I_n := \{a_n : \exists \sum_{i=n}^{\infty} a_i x^i \in I\}$. Then adapt the proof of the Hilbert basis theorem.
(4) $C^0(\mathbb{R})$, the ring of continuous functions $\mathbb{R} \to \mathbb{R}$ with pointwise operations.
(5) $\mathbb{R}[x]\Big/\left((x-1)^2 x\right)$.

[1] $R[[x]] = \{\sum_{i=0}^{\infty} a_i x^i : a_i \in R\}$, where multiplication and addition are defined formally, as what you think they should be. These are purely formal objects: there is no requirement for any kind of convergence.