EPFL - Fall 2024	Domenico Valloni
Rings and modules	**Exercises**
Sheet 1	19 September 2024

As always in the course, we fix the following notation:

- R is a ring,
- k is a field,
- if not specified, "module" means "left R-module", and
- M and N are modules.

Exercise 1. (1) A simple module is a module that has only trivial submodules. Show that any simple module is cyclic.

(2) Let $m \in M$ be an element. We define the annihilator of m by

$$\mathrm{Ann}_R(m) = \left\{ r \in R \;\middle|\; rm = 0 \right\}$$

We only write $\mathrm{Ann}(m)$ if it the base ring is clear from the context.

Show that $\mathrm{Ann}(m)$ is a left ideal of R and that the cyclic module Rm is isomorphic to the module $R/\mathrm{Ann}(m)$.

(3) Let M be a simple $k[x]$-module. Prove that $M \cong k[x]/(f)$ where f is an irreducible polynomial in $k[x]$ and (f) denotes the ideal generated by f.

(4) Which of the following $\mathbb{Z}$-modules are simple?
   (a) $\mathbb{Z}$
   (b) $\mathbb{Z}/6\mathbb{Z}$
   (c) $\mathbb{Z}/7\mathbb{Z}$

Exercise 2. Let R be a ring, M a left R-module and $m \in M$.

(1) In the previous exercise you proved that $\mathrm{Ann}(m)$ is a left ideal of R. Give an example to show that $\mathrm{Ann}(m)$ might *not* be a two sided ideal of R.

(2) Define the *annihilator* of M to be

$$\mathrm{Ann}_R(M) = \left\{ r \in R \;\middle|\; rM = 0 \right\} = \left\{ r \in R \;\middle|\; \forall m \in M \colon rm = 0 \right\}$$

Prove that $\mathrm{Ann}(M)$ is a two sided ideal of R.

(3) Let $\phi : S \to R$ be a surjective homomorphism of rings and M a module over S. Show that we can endow an R-module structure given by $r \cdot m = s \cdot m$ for any $s \in \phi^{-1}(r)$ and $m \in M$ if and only if $\ker \phi \subseteq \mathrm{Ann}(M)$.

(4) For example, let $S = k[x]$ and $M = k[x]$ (with the standard action). Then M/f^2M is a $k[x]/(f^2)$-module for any $0 \neq f \in k[x]$. In addition, if f is not invertible, then M/f^2M is not a $k[x]/(f)$-module.

Exercise 3. Answer the following questions. Provide an explanation by a proof or a counterexample.

(1) Suppose that R is a Noetherian ring. Let $S \subset R$ be a subring. Is it true that S is Noetherian?

(2) Let R be a commutative Artinian ring. Is every prime ideal of R maximal?

Exercise 4. Let $I \subseteq R$ be an ideal.

(1) Show that
$$IM = \left\{ \sum_{i=1}^{d} r_i m_i \;\middle|\; 1 \leq d \in \mathbb{Z},\ r_i \in I,\ m_i \in M \right\}$$
is an R-submodule of M.
(2) Show that M/IM is an R/I-module with scalar multiplication given by
$$(x+I)(y+IM) = xy + IM.$$

From now on, fix $R := k[x,y]$, M the R-submodule generated by the element $(x,y) \in R \oplus R =: N$, and let I be the maximal ideal $I = Rx + Ry$ of R. Note that $R/I \cong k$ via the homomorphism $R \to k$ that evaluates x and y to 0.

(3) Show that $M \subseteq IN$ and hence $I\left(N/M\right) = IN/M$ as R-submodules of N/M.
(4) Show that L/IL is a two dimensional vector-space over k, where $L = N/M$.

Now, we change a little bit our setup, and we redefine M:

(5) Let M be the submodule generated by the two elements $(x,0)$ and $(0,y)$ of $R \oplus R =: N$. Is $N/M \cong R$?
[Hint: look at $\mathrm{Ann}\left(N/M\right)$.]

Exercise 5. Let
$$0 \to M \to N \to N/M \to 0$$
be a short exact sequence of R-modules. For each of the following assertions either prove that the assertion holds or provide a counterexample.
(1) If M and N/M are finitely generated, then N is too.
(2) Conversely, if N is finitely generated, then N/M is finitely generated too.
(3) If N is finitely generated, then M is finitely generated too.

Exercise 6. (1) Let
$$0 \to M \to N \to N/M \to 0$$
be a short exact sequence of R-modules. For each of the following assertions either prove that the assertion holds or provide a counterexample.
- If N is free, then N/M is free.
- If N is free, then M is free.
- If M and N/M are free, then N is free.

(2) Let $R = \mathbb{Z}$. Is $\mathbb{Z}[x]/(x^2+1)\mathbb{Z}[x]$ a free R-module? How about $\mathbb{Z}[x]/(2x^2)\mathbb{Z}[x]$? Is $\mathbb{Q}$ a free R-module? Is it finitely generated?

Optional exercise. Not on the exam. Suggested if you are seriously interested in algebra.

Exercise 7. Let k be a field. In this exercise, we want to understand *differential operators* on $k[x]$. To this end, define the operator $\frac{\partial}{\partial x} \in \mathrm{End}_k(k[x])$ by the usual rule
$$\frac{\partial}{\partial x}(x^n) := nx^{n-1}.$$

Define also $x \in \mathrm{End}_k(k[x])$ defined by multiplication by x. Finally, define the subring $\mathcal{D} \subseteq \mathrm{End}_k(k[x])$ to be the sub-k-algebra generated by x and $\frac{\partial}{\partial x}$.

We will show that this non-commutative rung behaves very differently, whether we work in characteristic zero or in positive characteristic.

(1) Show that a basis of $\mathcal{D}$ as a k-vector space is given by the elements $x^i \left(\frac{\partial}{\partial x}\right)^j$, where $(i,j) \in \mathbb{N}^2$ if $\mathrm{char}\, k = 0$, and $i \in \mathbb{N}$ and $j \in \{0, 1, \ldots, p-1\}$ if $\mathrm{char}\, k = p > 0$.

(2) Now we change the perspective and consider a quotient of the free k-algebra on two generators $\mathcal{D}^{form} = k\langle u, v\rangle/(uv - vu - 1)$. Prove that in $\mathcal{D}^{form}$ we have the identity

$$uP(v) = \frac{\partial}{\partial v}P(v) + P(v)u$$

for all polynomials $P(v) \in k[v]$. Use this to prove that $\mathcal{D}^{form}$ is generated as a k-vector space by $\{v^j u^i \mid (i,j) \in \mathbb{N}^2\}$.

(3) Show that there are well defined ring homomorphisms ϕ and ψ from $\mathcal{D}^{form}$ to $\mathrm{End}_k(k[x])$, such that $\phi(u) = \frac{\partial}{\partial x}$ and $\phi(v) = x$, as well as $\psi(u) = x$ and $\psi(v) = -\frac{\partial}{\partial x}$. Show that ϕ and ψ are surjective onto $\mathcal{D}$, and define an isomorphism betwenn $\mathcal{D}$ and $\mathcal{D}^{form}$ if and only if $\mathrm{char}(k) = 0$.

(4) Show that $k[x]$ is simple as a left $\mathcal{D}$-module (with left $\mathcal{D}$-module structure given by the inclusion $\mathcal{D} \subset \mathrm{End}_k(k[x])$) in the case when $\mathrm{char}\, k = 0$.

(5) Determine the left submodules of $k[x]$ as a $\mathcal{D}$-module when $\mathrm{char}\, k = 2$.

Remark 0.1. For any field k, regardless of the characteristic, one can define the notion of a differential operator. In characteristic zero, this definition agrees with our $\mathcal{D}$ above, but this is not the case in characteristic $p > 0$. In other words, there are more exotic differential operators in positive characteristic, not spanned by $\frac{\partial}{\partial x}$ and x! An example is what we usually write

$$\delta := \frac{1}{p}\left(\frac{\partial}{\partial x}\right)^p,$$

defined as

$$\delta(x^n) = \frac{n(n-1)\ldots(n-p+1)}{p}x^{n-p},$$

where we note that, even if we work in characteristic p, the element

$$\frac{n(n-1)\ldots(n-p+1)}{p}$$

makes sense in k, since it lives in $\mathbb{Z}$ and there is a natural map $\mathbb{Z} \to k$

With this more general notion of differential operators, $k[x]$ becomes again simple (in any characteristic!).