EPFL - Fall 2024	Domenico Valloni
Rings and modules	**Exercises**
Sheet 12	12 December 2023

The goal of this exercise is to see that the statement of Exercise 8 is wrong without the algebraically closed assumption.

Exercise 1. (1) Let $R \to S$ be a morphism of commutative rings (thus making S an R-algebra), and let I be an ideal of $R[x_1, \dots, x_n]$. Then we have an isomorphism of S-algebras

$$R[x_1, \dots, x_n]\big/I \otimes_R S \cong S[x_1, \dots, x_n]\big/(I)$$

[*Hint: First show it for $I = 0$, and then deduce the general case using right exactness of the tensor product. The case $I = 0$ can be handled by a direct computation, or by showing that both sides satisfy the same universal property.*]

(2) Show that

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$$

and hence it is not a domain (but it is nevertheless reduced!)

(3) Show that

$$\mathbb{F}_p(x) \otimes_{\mathbb{F}_p(x^p)} \mathbb{F}_p(x) \cong \mathbb{F}_p(x)[t]\big/(t-x)^p$$

which is not even reduced.

Exercise 2. Let M be an A-module, and let $\mathfrak{a}$ be an ideal in A. Show that the following are equivalent:

(1) $M = 0$,
(2) $M_{\mathfrak{p}} = 0$, for every prime ideal $\mathfrak{p} \subseteq A$,
(3) $M_{\mathfrak{m}} = 0$, for every maximal ideal $\mathfrak{m} \subseteq A$.

Moreover, suppose that M is a finitely generated A-module, under this assumption prove that $M = \mathfrak{a}M$ if and only if $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ satisfying $\mathfrak{a} \subseteq \mathfrak{m}$.
[*Hint/Remark: Although the exercise can be solved without directly proving the implication* $(3) \Rightarrow (2)$, *it is highly instructive for anyone who thinks about studying more commutative algebra/algebraic geometry, to think through the* $(3) \Rightarrow (2)$ *implication using Exercise 7.*]

Exercise 3. Let $R = F[x]$, where F is a field.

(1) If F is algebraically closed, then show that for every prime ideal $\mathfrak{p}$ of R, either $R_{\mathfrak{p}} \cong F(x)$ or $R_{\mathfrak{p}} \cong F[x]_{(x)}$, where these isomorphisms are isomorphisms of F-algebras. Show that the above two cases are not isomorphic.
(2) If $F = \mathbb{R}$, then show that up to ring isomorphism there are three possibilities for $R_{\mathfrak{p}}$, where $\mathfrak{p}$ is a prime ideal of $F[x]$.
[*Hint:* To tell the three cases apart, consider the residue field, to show that there are only three cases, apply linear transformations to x.]
(3) Show that if F is algebraically closed, then $F[x,y]$ has infinitely many prime ideals $\mathfrak{p}$ for which $F[x,y]_{\mathfrak{p}}$ are pairwise non-isomorphic F-algebras. For this, you can use the following theorem of algebraic geometry:

Theorem. *There exists a sequence of irreducible polynomials $(f_d)_{d\in\mathbb{N}\setminus\{0,2\}}$ in $F[x,y]$ such that f_d is of degree d and such that the fields $\mathrm{Frac}\left(F[x,y]\big/(f_d)\right)$ are pairwise non-isomorphic as F-algebras.*

Exercise 4. Let F be an algebraically closed field.

(1) List the prime ideals of $R = {}^{F[x,y]}\big/_{(xy)}$.
[*Hint:* Consider the implications of a containment $xy \in \mathfrak{p}$, for a prime ideal $\mathfrak{p}$. Consider the projections $R \to {}^{R}\big/_{(x)}$ and $R \to {}^{R}\big/_{(y)}$ and use that you know the prime ideals of $F[y]$ and $F[x]$.]

(2) Show that for all prime ideals $\mathfrak{p}$ of R, $R_{\mathfrak{p}}$ falls into three cases up to F-algebra isomorphism, one which is a field, one which is a domain but not a field and one which is not a domain.

Exercise 5. Let R be a commutative ring.

(1) Let $T \subseteq R$ a multiplicatively closed subset of R. Let $\mathfrak{q}$ be a prime ideal of $T^{-1}R$. Let $\mathfrak{q}^c$ be the contraction of q under $R \to T^{-1}R$. Prove that $\operatorname{ht}(\mathfrak{q}) = \operatorname{ht}(\mathfrak{q}^c)$.

(2) Let $\mathfrak{p}$ be a prime ideal of R. Prove that $\operatorname{ht}(\mathfrak{p}) = \dim R_{\mathfrak{p}}$.

Exercise 6. Let $S \to R$ be a morphism of rings. Show that a prime ideal $\mathfrak{p}$ of S is the contraction of a prime ideal of R if and only if $\mathfrak{p}^{ec} = \mathfrak{p}$.
[*Hint:* For one direction use ideas from the proof of Going-Up Theorem (Proposition 9.4.2 of the lecture notes).]

Exercise 7. Let R be a ring, let M be an R-module and let $T, S \subseteq R$ be two multiplicatively closed subsets of R. Define $ST := \{st \mid s \in S, t \in T\}$ and $\tilde{S} := \{s/1 \mid s \in S\} \subseteq T^{-1}R$.

(1) Show that ST and $\tilde{S}$ are multiplicatively closed subsets of R resp. $T^{-1}R$.

(2) Show that there exists a ring morphism $\tilde{S}^{-1}(T^{-1}R) \to (ST)^{-1}R$ sending $(r/t)/(s/1) \in \tilde{S}^{-1}(T^{-1}R)$ to $r/(st) \in (ST)^{-1}R$. Show further that this is an isomorphism.

(3) Show that $\tilde{S}^{-1}(T^{-1}M)$ and $(ST)^{-1}M$ are isomorphic as $(ST)^{-1}R$-modules, where the $(ST)^{-1}R$-module structure of $\tilde{S}^{-1}(T^{-1}M)$ is provided via the isomorphism of the previous point.

(4) Show that if $T \subseteq S$ then $ST = S$, and formulate the results of points (2) and (3) in this case.

Exercise 8. In Exercise 6 of sheet 10, we saw how to construct the tensor product of two R-algebras. The goal is to show the following result:

Proposition 0.1. *Let k be an algebraically closed field, and let R, S two finitely generated k-algebras which are domains. Then $R \otimes_k S$ is again a domain.*

Recall the following important facts from the course:

- Nullstellensatz (Theorem 6.5.4 from the notes)
- For any finitely generated k-algebra T and any maximal ideal $\mathfrak{m}$, the composition $k \to T \to T/\mathfrak{m}$ is an isomorphism (see the proof of the weak Nullstelensatz, which is Theorem 6.2.2 in the notes).

Proceed as follows:

(1) Let T be a finitely generated k-algebra which is a domain, and let $a_1, \dots a_s \in T$ be non-zero. Show that there is a maximal ideal $\mathfrak{m}$ of T such that $a_i \notin \mathfrak{m}$ for all i.
[*Hint*: write T as a quotient of a polynomial ring, and use Nullstellensatz.]

(2) Show that any element in $R \otimes_k S$ can be written as

$$\sum_i a_i \otimes b_i$$

with the b_i's linearly independent over k.

(3) Assume that

$$\left(\sum_i a_i \otimes b_i\right) \cdot \left(\sum_j a'_j \otimes b'_j\right) = 0$$

where both families $(b_i)_i$ and $(b'_j)_j$ are linearly independent. Let $\mathfrak{m}$ be a maximal ideal not containing any of the a_i, a'_j.
Show by applying the ring map

$$R \otimes_k S \to R/\mathfrak{m} \otimes_k S \cong S$$

that one of the factors must be zero, and hence conclude that $R \otimes_k S$ is a domain.

Remark 0.2. By the first exercise, the algebraic closedness condition is crucial in the above.