EPFL - Fall 2024	Domenico Valloni
Rings and modules	**Exercises**
Sheet 11	05 December 2024

In this exercise sheet, all rings are commutative. The exercise marked by ♠ is this week's bonus exercise. You can hand in your LaTeX-solutions on Moodle until Wednesday the 18th of December at 6pm sharp.

General definitions around the notion of a functor

Let us see how the construction of the Ext-modules can be adapted to an additive right exact covariant functor.

Definition 1. Given R a commutative ring, we set Mod_R to be the category of R-modules.
A covariant functor $F : \mathrm{Mod}_R \to \mathrm{Mod}_S$ is said to be *additive* if $F(0) = 0$ (for both the 0 map and the 0 object) and for all R-modules M, N, the natural map

$$F(M \oplus N) \to F(M) \oplus F(N)$$

(induced by the projections) is an isomorphism.
An additive covariant functor is said to be *right exact* is for any short exact sequence

$$M \to N \to P \to 0$$

the sequence

$$F(M) \to F(N) \to F(P) \to 0$$

is exact.

Definition 2. Let F be an additive right exact covariant functor. For an R-module M, perform the following construction.

(1) Take a projective resoultion $P_\bullet \to M$ of M.
(2) Apply F to the projective resolution to obtain a chain complex $F(P_\bullet)$ given by

$$\cdots \to F(P_1) \to F(P_0) \to 0.$$

(3) Define $L_iF(M) = H_i(F(P_\bullet))$, i.e. $L_iF(M)$ is the i-th homology group of the chain complex $F(P_\bullet)$.

One can show that $L_iF(M)$ doesn't depend on the projective resolution, so that it gives a well defined object. In a similar way as for the Ext-modules, one can also construct $L_iF(f) : L_iF(M) \to L_iF(M')$ for any R-module homomorphism $f : M \to M'$, and show that this turns L_iF into a covariant functor. It is called the i-th left derived functor of F.

As mentioned in Remark 9.1.7 of the printed course notes, the Tor-functors are the left derived functors of $- \otimes_R N$. In formulas, $\mathrm{Tor}_i^R(M, N) = L_i(- \otimes_R N)(M)$.

One can prove the corresponding statement of Theorem 5.5.6 also for the left derived functors of a covariant functor:

Theorem. *Let F be a right exact covariant functor and let $0 \to K \to M \to L \to 0$ be a short exact sequence. Then there exists a long exact sequence*

$$\cdots \to L_2F(K) \to L_1F(K) \to L_1F(M) \to L_1F(L) \to FK \to FM \to FL \to 0$$

Exercise 1. Let R be a ring and let M, N be R-modules. Prove that $\mathrm{Tor}_0^R(M,N) \cong M \otimes_R N$. [*Hint:* Try to adapt the proof of Proposition 5.3.8 in the printed course notes.]

Exercise 2. Let R be a ring and N an R-module. We say that N is *flat* if for every short exact sequence of R-modules

$$0 \to M \to M' \to M'' \to 0$$

the sequence

$$0 \to M \otimes_R N \to M' \otimes_R N \to M'' \otimes_R N \to 0$$

is exact. Prove that the following are equivalent:

(1) N is flat,
(2) $\mathrm{Tor}_i^R(M,N) = 0$ for every R-module M and every $i > 0$,
(3) $\mathrm{Tor}_1^R(M,N) = 0$ for every R-module M.

[*Hint:* For (1)⇒(2) take a free resolution of M and tensor it with N to compute the Tor-functors. For (3)⇒(1) use the long exact sequence for left derived functors.]

Exercise 3. Let $R = k[x,y]$ where k is a field. Consider the R-modules $M := (x,y)$ (i.e. the ideal generated by x and y) and $N := R/M$.

(1) Compute $\mathrm{Tor}_i^R(M,N)$ for all integers $i \geq 0$.
[*Hint:* Use the definition.]
(2) Is N flat?
(3) Compute $\mathrm{Tor}_i^R(N,N)$ for all integers $i \geq 0$.
[*Hint:* Use the long exact sequence.]

Exercise 4. Let R be a ring.

(1) Prove that free R-modules are flat.
(2) Prove that projective R-modules are flat.
[*Hint:* Use the characterization of projective modules as direct summands of free modules.]
(3) Assume that R is an integral domain. Determine for which ideals I of R the R-module R/I is flat.

Remark 0.1. There exists a partial converse of (2): a flat finitely generated module over a Noetherian ring is projective.

The finite generation hypothesis is very important, as the $\mathbb{Z}$-module $\mathbb{Q}$ is flat but not projective. There are also counter-examples in the Non-noetherian case.

Exercise 5. Let R be a ring containing a multiplicatively closed subset T, and let M be an R-module. Show that there is an isomorphism of R-modules

$$T^{-1}M \cong T^{-1}R \otimes_R M.$$

Further show that this is an isomorphism of $T^{-1}R$-modules.
[*Remark:* The right hand side naturally has the structure of a $T^{-1}R$-module by point (1) of Exercise 6 on Sheet 10.]

Exercise 6. Let R be a ring with multiplicative subset T, and suppose that L, M and N are R-modules.

(1) Show that if there is an R-module homomorphism $f : M \to N$ then there is a natural $T^{-1}R$-module homomorphism $f_T : T^{-1}M \to T^{-1}N$.
(2) Show that there is an isomorphism of $T^{-1}R$-modules $T^{-1}(M \oplus N) \cong (T^{-1}M) \oplus (T^{-1}N)$.
(3) Suppose there is an exact sequence

$$0 \to L \to M \to N \to 0.$$

Prove that the sequence

$$0 \to T^{-1}L \to T^{-1}M \to T^{-1}N \to 0$$

is also exact. Deduce that if $L \subset M$ is a sub R-module, then $T^{-1}\left(M/L\right) \cong T^{-1}M / T^{-1}L$ and that localization by T is an exact functor of R-modules and that $T^{-1}R$ is a flat R-module.
(4) Let $\mathfrak{p}$ be a prime ideal of R. Show that there is an isomorphism of rings $\operatorname{Frac}\left(R/\mathfrak{p}\right) \cong R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.
[*Remark:* For a local ring A with maximal ideal $\mathfrak{m}$ we call $A/\mathfrak{m}$ the residue field of A.]

Exercise 7. Let R be a ring, let S be a multiplicatively closed subset, and let M and N be R-modules. Show that for all $i \geq 0$,

$$S^{-1}\operatorname{Tor}_i^R(M,N) \cong \operatorname{Tor}_i^{S^{-1}R}(S^{-1}M, S^{-1}N).$$

If furthermore R is Noetherian and M is finitely generated, then also

$$S^{-1}\operatorname{Ext}^i_R(M,N) \cong \operatorname{Ext}^i_{S^{-1}R}(S^{-1}M, S^{-1}N).$$

Given a commutative ring R, a prime ideal $\mathfrak{p}$ of R is said to be *minimal* it if contains no strict prime ideal. For example, if R is a domain, then (0) is the only minimal prime.

For the following exercise, you may use without proof the following result, which we will see later (although its proof does not require any sophisticated tool):

Proposition 0.2. *Let R be a Noetherian ring. Then R contains at most finitely many minimal prime ideals.*

Exercise 8. ♠ As we have seen in Exercise 3.2 of sheet 1, an Artinian ring has dimension zero. The goal of this exercise is to show the converse. Let R be a zero-dimensional Noetherian ring. Proceed as follows:

(1) Assume that R is local with maximal ideal $\mathfrak{m}$. Show that $\mathfrak{m}^n = 0$ for some $n \geq 1$.
(2) With the same assumptions, conclude that R is Artinian.

(3) Use the local case to show that R is Artinian in general.
Hint: Use the proposition above and Exercise 2 of sheet 12.
(4) Find an example of a zero-dimensional ring which is not Artinian.