EPFL - Fall 2024	Domenico Valloni
Rings and modules	**Exercises**
Sheet 10	28 November 2024

Exercise 1. Let G be a finite group, R an integrally closed domain, K the fraction field of R and let G act on K by (ring) automorphisms such that R is stable under this action, i.e. $g \cdot r \in R$ for all $g \in G$ and $r \in R$. Let $L := K^G$ be the fixed field of the action and set $S := L \cap R$. In this exercise we show that S is also integrally closed.

(1) Show that each element of K can be written in the form $\frac{a}{b}$, where $a \in R$ and $b \in S$.
(2) Show that L is the fraction field of S.
(3) Show that S is integrally closed.
(4) Show that $\mathbb{C}[x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n] \subseteq \mathbb{C}[x, y]$ is integrally closed.
[*Hint:* Show that there is automorphism of $\mathbb{C}(x, y)$ that sends x to $e^{2\pi i/n}x$ and y to $e^{2\pi i/n}y$.]

Exercise 2. Let k be a field. For the following finitely generated k-algebras R, find a sub-algebra $S \subseteq R$ such that $S \subseteq R$ is integral and S is isomorphic to a polynomial ring:

(1) $R = k[x, y]\big/(xy - 1)$;
(2) $R = k[x_1, x_2, x_3, y_1, y_2, y_3]\big/(x_1x_2x_3 + y_1y_2y_3)$;

Exercise 3. Show that the ring

$$k[x, y, z]\big/(y^3 + y^2x^2 + yx^2 + x^3z)$$

is a domain, and compute its integral closure.

General definitions around the notion of a functor

Now let us study the tensor product, and in particular also about some of its functorial properties. As in the course we only saw the concept of a functor in specific situations (i.e. the Hom- and the Ext-functors), we will recall here everything which is needed to develop a similar treatment for the tensor product. You can use everything in these grey boxes without proof.

Definition 1. We say that $F : \{R\text{-modules}\} \to \{R\text{-modules}\}$ is a covariant functor if for every R-module M we have an R-module FM and for every R-module homomorphism $f : M \to M'$ we have an R-module homomorphism $F(f) : FM \to FM'$ such that

(1) $F(\mathrm{id}_M) = \mathrm{id}_{FM}$ for all R-modules M
(2) $F(f' \circ f) = F(f') \circ F(f)$ for all R-module homomorphisms $f : M \to M'$ and $f' : M' \to M''$.

Recall that in section 5.2 of the printed course notes we called $\mathrm{Hom}_R(-, N)$ a contravariant functor. The difference between a covariant functor and a contravariant functor is that a covariant functor preserves the direction of arrows, while a contravariant functor flips the direction of arrows (and condition (2) in the above definition is replaced by the appropriate equation).

Definition 2. We say that a covariant functor F is *right exact* if for every short exact sequence of R-modules $0 \to M \to M' \to M'' \to 0$ the sequence $FM \to FM' \to FM'' \to 0$ is exact.

Recall that in Lemma 5.2.2 of the printed course notes we proved that $\mathrm{Hom}_R(-, N)$ is a left exact contravariant functor, and on Exercise 1 of Sheet 4 we saw that $\mathrm{Hom}_R(M, -)$ is a left exact covariant functor.

Exercise 4. Let R be a ring. Let M, N be R-modules and I an ideal of R. Prove that there are isomorphisms of R-modules $M \otimes_R N \cong N \otimes_R M$ and $M \otimes_R \left({}^{R}\!/_{I}\right) \cong {}^{M}\!/_{IM}$.

Exercise 5. Let R be a ring, and M, N and P be R-modules. Show that there exists a natural bijection

$$\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)).$$

Use this to prove that

$$- \otimes_R N : \{R\text{-modules}\} \to \{R\text{-modules}\}, \quad A \mapsto A \otimes_R N$$

is a right exact covariant functor.

[*Hint:* Show that a sequence of the form

$$\mathcal{S} := A \to B \to C \to 0$$

is exact if and only if $\mathrm{Hom}(\mathcal{S}, P)$ is exact for all modules P]

Remark 0.1. The hint above is a particular phenomenon of a general (not complicated) result called Yoneda's lemma, which can be read as "a module is entirely determined by how it maps to other modules". A precise way to say it is that if M and N are two modules such that there is a natural isomorphism (in the sense of category theory) $\mathrm{Hom}(M, P) \cong \mathrm{Hom}(N, P)$ for all P, then $M \cong N$. It is always good (and sometimes even useful!) to keep this philosophy in mind.
In fact, this lemma holds for arbitrary categories, not only for modules.

Exercise 6. Let A be a ring, with A-algebras B and C and an A-module M. Show that:

(1) $B \otimes_A M$ naturally has the structure of a B-module,
(2) $B \otimes_A C$ naturally has the structure of an A-algebra,
(3) $B \otimes_A B$ naturally has a ring morphism to B.

Exercise 7. Prove the following assertions:

(1) Let R be a commutative ring, and let M_1 and M_2 be free R-modules with bases $\{e_1, ..., e_m\}$ and $\{f_1, ..., f_n\}$ respectively. Show that a basis of $M_1 \otimes_R M_2$ is given by $\{e_i \otimes f_j\}_{\substack{1 \le i \le m \\ 1 \le j \le n}}$.
(2) Hence show that the element $e_1 \otimes f_2 + e_2 \otimes f_1$ cannot be written as $u \otimes v$ for any $u \in M_1$ and $v \in M_2$.

Exercise 8. We will define the exterior product of a module. This construction is especially important, for example in differential/algebraic geometry when one considers differential forms.

Let R be a commutative ring, and let M be an R-module. For any $n > 0$, define $T^n(M) := M \otimes_R \cdots \otimes_R M$ (n times). We also set $T^0(M) := R$. For any $n \geq 0$, we define $\bigwedge^n M$ as the quotient of $T^n M$ by the submodule I generated by elements of the form

$$m_1 \otimes \cdots \otimes m_n,$$

with $m_i = m_j$ for some $i \neq j$. The image of $m_1 \otimes \cdots \otimes m_n$ in $\bigwedge^n M$ is denoted $m_1 \wedge \cdots \wedge m_n$.

Note that if $f\colon M \to N$ is a morphism of R-modules, then it naturally induced a morphism $T^n(f)\colon T^n(M) \to T^n(N)$ of R-modules (apply f to each tensor), and passes to the quotient $\bigwedge^n f\colon \bigwedge^n M \to \bigwedge^n N$.

From now on, assume that M is free of finite rank $r \geq 1$, with basis $\mathcal{B} = \{e_1, \ldots, e_r\}$.

- Show that $\bigwedge^r M$ is free with basis $e_1 \wedge \cdots \wedge e_r$, and that $\bigwedge^l M = 0$ for any $l > r$.
- Show that for $0 \leq i \leq r$, $\bigwedge^i M$ is free of rank $\binom{r}{i}$.

  Hint: First find a the appropriate number of generators. To show that it is a basis (i.e. the linear independance), wedge it by an appropriate element to get something in $\bigwedge^r M$, where you know an explicit basis.
- Fix the isomorphism $\theta\colon \bigwedge^r M \to R$ corresponding to the basis found in the first point. Let $f\colon M \to M$ be an endomorphism, corresponding to a matrix $A \in M_{r\times r}(R)$ (with respect to $\mathcal{B}$). Show that the diagram

$$\begin{tikzcd} \bigwedge^r M \arrow[r, "\bigwedge^r f"] \arrow[d, "\theta"'] & \bigwedge^r M \arrow[d, "\theta"] \\ R \arrow[r, "\cdot \det(A)"] & R \end{tikzcd}$$

  commutes.
- Use the above to give a new proof that if A and B are two $r \times r$-matrices, then $\det(AB) = \det(A)\det(B)$.

  Hint: $\bigwedge$ is functorial.