December 2020

Practice Problem Set

Exercise 1. *For each of the following statements, determine if it is true or false. Justify your answer by a proof or a counter-example.*

(a) *There are no nontrivial zero divisors in a field.*

(b) *Every element of a field is irreducible.*

(c) *In a ring R the element 1_R is always irreducible.*

(d) *Let K be a field. The polynomial ring $K[X]$ is a principal ideal domain.*

(e) *In a principal ideal domain any irreducible element generates a maximal ideal.*

(f) *The ring $\mathbb{Z}$ is a principal ideal domain.*

(g) *The ring $\mathbb{Z}[X]$ is a principal ideal domain.*

(h) *In the ring $\mathbb{Z}$ any irreducible element is positive.*

Exercise 2. *Consider the ideals $I = 24\mathbb{Z}$, $J = 30\mathbb{Z}$, $K = 9\mathbb{Z}$, $L = 3\mathbb{Z}$ and $M = 7\mathbb{Z}$ in the ring $\mathbb{Z}$. For each of the following ideals find $m \in \mathbb{Z}$ such that the given ideal is equal to $(m) = m\mathbb{Z} \in \mathbb{Z}$.*

$$I + J,\ I \cap J,\ I + J + L,\ I + M,\ I \cap L,\quad J + I + M,\ ,IJ \cap K.$$

Do the same for the ring $\mathbb{R}[X]$ and the ideals $L = (X-1)$, $K = (X+1)$, $M = (X^2+1)$, $I = (X^2-X)$ et $J = (X^4-1)$.

Exercise 3. *Show that the ideal $I = (2, X)$ in the ring $\mathbb{Z}[X]$ is not principal. Describe the ideal $I = (2, X)$ in the ring $\mathbb{Q}[X]$.*

Exercise 4. *Let A and B be two rings, $U(A) \subset A$ and $U(B) \subset B$ the groups of invertible elements and $\Phi : A \to B$ a ring homomorphism.*

(a) *Show that the set $\Phi(U(A))$ is a subgroup of $U(B)$.*

(b) *Suppose that $\Phi : A \to B$ is surjective. Is it always true that $\Phi(U(A)) = U(B)$? Hint: consider the case $A = \mathbb{Z}$, $B = \mathbb{Z}/7\mathbb{Z}$ and $\Phi(k) = [k]_7$ for all $k \in \mathbb{Z}$.*

Exercise 5. *Let $\Phi_1 : \mathbb{Z}/11\mathbb{Z} \to A$ and $\Phi_2 : \mathbb{Z}/15\mathbb{Z} \to B$ be ring homomorphisms. What can be the number of elements in the image of Φ_1 (respectively Φ_2)?*

Exercise 6. *(a) Find the characteristic of the polynomial rings $\mathbb{Z}[x]$, $\mathbb{R}[x]$ et $\mathbb{F}_p[x]$.*

(b) *Find the order and the characteristic of $\mathbb{F}_2[x]/I$, where I is generated by the ideal $x^3 - 1$.*

Exercise 7. *(a) Find the monic greatest common divisor of the polynomials $2x^3 - 11x^2 + 2x - 11$ and $x^2 + 1$ in $\mathbb{Q}[x]$.*

(b) *Are the polynomials $h_1(x) = x^3 - 2x^2 - x - 18$ and $h_2(x) = x^2 - 5x - 6$ coprime in $\mathbb{Q}[x]$?*

(c) *Which of the polynomials $f_1(x) = x^3 + 1$, $f_2(x) = x^3 + x^2 + 1$, $f_3(x) = x^3 + x^2 + x + 1$ are irreducible in $\mathbb{F}_2[x]$? Give the factorization into irreducible factors for those that are not irreducible.*

(d) *Are the polynomials $g_1(x) = x^2 - 2$ and $g_2(x) = x^2 - 3$ irreducible in $\mathbb{Q}[x]$? in $\mathbb{F}_{11}[x]$?*

Exercise 8. *(a) Show that the fields $\mathbb{Q}[\sqrt{3}]$ and $\mathbb{Q}[\sqrt{5}]$ are not isomorphic.*

(b) *Show that the rings $\mathbb{F}_5[x]/(x^2 - 2)$ are $\mathbb{F}_5[x]/(x^2 - 3)$ are fields. Are they isomorphic?*

(c) *Find an explicit isormophism between the fields $\mathbb{R}[x]/(x^2 - 2x + 2)$ and $\mathbb{R}[x]/(x^2 + 1)$.*

Exercise 9. *Give examples of fields of* 25 *and* 27 *elements.*

Exercise 10. *(a) Show that the polynomial* $X^4 + X + 1$ *is irreducible over* $\mathbb{F}_2$.

(b) Let I *be the ideal* $(X^4 + X + 1)$ *in* $\mathbb{F}_2[X]$. *Find the number of elements in the field* $\mathbb{F}_2[X]/I$ *and the inverse of the element* $g = [X + 1]_I$.

(c) List all irreducible polynomials of degree 4 *over* $\mathbb{F}_2$.

Exercise 11. *Let* S_{2k} *denote the symmetric group of permutation of* $2k$ *elements.*

(a) Prove that S_{2k} *contains an abelian subgroup of order* 2^k *such that all of its elements except* 1 *have order* 2.

(b) Determine the decomposition of this subgroup as a direct product of cyclic groups with orders given by the elementary divisors.

Exercise 12. *Let* S_n *denote the symmetric group of permutation of* n *elements, and suppose that* $n \geq k_1 + k_2 + \ldots + k_r$ *for some integers* $k_i \geq 2$. *Let* $t \in S_n$ *be a product of disjoint cycles of lengths* $k_1, k_2, \ldots k_r$,

$$t = \pi_{k_1} \pi_{k_2} \ldots \pi_{k_r}.$$

Find the order of the element t *in* S_n.

Exercise 13. *Let* S_5 *denote the symmetric group of permutation of* 5 *elements. Let* $a = (135)(24) \in S_5$, *and* $b = (134)(24) \in S_5$.

(a) Find the order of a *and* b *in* S_5.

(b) Let $A = \langle a \rangle \subset S_5$ *and* $B = \langle b \rangle \subset S_5$ *be the subgroups generated by these elements in* S_5. *Find the orbit of the element* 1 *with respect to the action of* A *and* B, *and its stabilizer subgroup in* A *and* B, *and show how the Orbit-Stabilizer theorem works in these cases.*

Exercise 14. *What is the smallest symmetric group that contains a subgroup isomorphic to*
(a) C_{60},
(b) C_{110},
(c) C_{27} *?*