November 18, 2024

Problem Set 9

Exercise 1. Let R be a ring and I, J (two-sided) ideals in R. Show that $I+J$, $I \cap J$ and IJ are ideals in R. Compute the ideals $I+J$, $I \cap J$, IJ if $I = 30\mathbb{Z}$ and $J = 24\mathbb{Z}$ in the ring of integers $\mathbb{Z}$.

Exercise 2. (a) Consider the ring $\mathbb{Z}[X]$ of polynomials in one variable with integer coefficients. Let $I = (2, X)$ be the ideal generated by elements 2 and X. Describe the polynomials in this ideal.

(b) Let A be a commutative ring and I, J ideals in A. Show also that the set $I \star J = \{ab \mid a \in I, b \in J\}$ is not in general an ideal in A, by providing the counter-example of the set $I \star J \subset \mathbb{Z}[X]$, where $I = J = (2, X) \subset \mathbb{Z}[X]$.

(c) Let $I = (m) \subset \mathbb{Z}$ and $J = (n) \subset \mathbb{Z}$. Is $I \star J$ an ideal in $\mathbb{Z}$?

Exercise 3. Let I be the smallest ideal in $\mathbb{Z}$ containing $\{392, 224, 168\}$. Find $d \in \mathbb{N}$ such that $I = (d) \subset \mathbb{Z}$.

Exercise 4. Let $I = ((x^2+x-6)) \subset \mathbb{R}[x]$ be the ideal of polynomials divisible by (x^2+x-6), and $J = ((x^2-x-2)) \subset \mathbb{R}[x]$ the ideal of polynomials divisible by (x^2-x-2). Describe the ideals $I, J, I \cap J, I \cdot J, I + J$.

Exercise 5. Let A be a commutative ring and $N \subset A$ a subset of all nilpotent elements in A:

$$N = \{x \in A : \ \exists n \in \mathbb{N} : \ x^n = 0\}.$$

Show that $N \subset A$ is an ideal. This ideal is called the *nilradical* of the ring. Show that the quotient ring A/N has no nonzero nilpotent elements.