November 11, 2024

Problem Set 8

Exercise 1. (a) Let m be an integer. Show that the ring modulo m, $\mathbb{Z}/m\mathbb{Z}$, is an integral domain if and only if m is a prime.

(b) Find all zero divisors and all invertible elements (units) in $\mathbb{Z}/30\mathbb{Z}$.

(c) Show that $[a]_m \in \mathbb{Z}/m\mathbb{Z}$ is invertible if and only if it is not a zero divisor.

Exercise 2. (a) Consider the set S of all polynomials with real coefficients of degree up to 3 with the usual addition and multiplication of polynomials. Is it a ring?

(b) Consider the ring $\mathbb{Z}[X]$ of all polynomials with integer coefficients. Check that it is a ring. Is it an integral domain?

(c) Let $R = \mathbb{Z}/4\mathbb{Z}$, and consider the ring $R[X]$ of polynomials with coefficients in R. Is it an integral domain? Justify your answer.

Exercise 3. Let $C[0,1]$ denote the ring of continuous real functions on the interval $[0,1]$.

(a) Let $f \in C[0,1]$ be such that the set $\{x : f(x) = 0\}$ contains a closed interval $[a,b] \subset [0,1]$ of positive length $b - a > 0$. Show that f is a zero divisor in $C[0,1]$.

(b) What are the invertible elements in the ring $C[0,1]$?

Exercise 4. (a) Show that a finite integral domain is a field.

(b) Find an example of a commutative finite ring that is not an integral domain.

(c) Find an example of an integral domain that is not a field.

Exercise 5. Let $C[0,1]$ be the ring of continuous functions on the interval $[0,1]$. Let S be a closed subset of $[0,1]$ and set $I_S = \{f \in C[0,1] \ : \ f(x) = 0 \ \text{ for all } \ x \in S\}$.

(a) Show that I_S is an ideal in $C[0,1]$.

(b) If $S_1 = [0, \frac{1}{2}]$, $S_2 = [\frac{1}{2}, 1]$, $S_3 = \{\frac{1}{3}\}$, $S_4 = \{\frac{2}{3}\}$, describe the ideals $I_{S_1} \cap I_{S_2}$, $I_{S_1} \cdot I_{S_2}$, $I_{S_1} + I_{S_2}$, $I_{S_3} \cap I_{S_4}$, $I_{S_3} \cdot I_{S_4}$, and $I_{S_3} + I_{S_4}$.