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Problem Set 6 Solutions

Exercise 1. Consider the permutation with the cycle decomposition $g = (12)(345)$ in the symmetric group S_5. Let $H = \langle g \rangle \subset S_5$ be the subgroup generated by this element.

(a) Find the order of H.

(b) Find the stabilizer subgroup in H of the element 1, and the orbit of 1 under the action of H.

(c) Find the stabilizer subgroup in H of the element 4, and the orbit of 4 under the action of H.

(d) In both cases, check the formula
$$[H : \mathrm{Stab}(x)] = |\mathrm{Orb}(x)|.$$

Solution 1. The order of g is 6, the product of the orders of the commuting cycles (12) and (345) (cf. Exercise 3(c), Problem Set 3). This can also be checked by a direct computation of the action of powers of g on the set of 5 elements.
(b) $\mathrm{Stab}(1) = \{1, (345), (354)\}$, and $\mathrm{Orb}(1) = \{1, 2\}$. We have $6 : 3 = 2$.
(c) $\mathrm{Stab}(4) = \{1, (12)\}$, and $\mathrm{Orb}(4) = \{3, 4, 5\}$. We have $6 : 2 = 3$.

Exercise 2. (a) Recall that for any $\pi \in S_n$ and any cycle $c \in S_n$, the element $\pi c \pi^{-1}$ is the cycle obtained by replacing each integer i in the cycle c with the integer $\pi(i)$. Use this property to show that $g_1, g_2 \in S_n$ are conjugate if and only if they decompose as a product of disjoint cycles of the same lengths.

(b) The orbits under the conjugation action are called the conjugacy classes of a group. Describe the conjugacy classes in the group S_5.

(c) Recall that the class equation of a finite group is $|G| = |Z| + \sum_{i=1}^{m} |C_i|$, where Z is the center of G (the set of all one-element conjugacy classes), and $\{C_i\}_{i=1}^{m}$ is the set of all nontrivial conjugacy classes of G. Count the number of elements in each conjugacy class and write the class equation for S_5.

Solution 2. (a) Let $x \in S_n$ and $x = c_1 c_2 \dots c_k$ the cycle decomposition of x. Then for $g \in S_n$,
$$gxg^{-1} = gc_1g^{-1}gc_2g^{-1} \dots gc_kg^{-1} = u_1u_2 \dots u_k,$$
where for each $i = 1 \dots k$ the cycle u_i is obtained from the cycle c_i by replacing each integer i with $g(i)$. Therefore, any element conjugate to x has the cycle decomposition with cycles of the same lengths. Now, suppose $x = c_1 \dots c_k$ and $y = u_1 \dots u_k$ are two permutations such that $|c_i| = |u_i|$ for all $i = 1 \dots k$. Let g be the permutation that sends the ordered set of integers in the cycle decomposition of x to the ordered set of integers in the cycle decomposition of y. Then, using the hint, we see that $gxg^{-1} = y$. Therefore all elements that decompose as a product of disjoint cycles of the same lengths are conjugate in S_n.

(b) According to (a), the conjugacy classes correspond to the products of disjoint cycles of different lengths. In S_5 we have:

- 5-cycles
- 4-cycles
- 3-cycles
- 2-cycles
- a product of two disjoint cycles of length 3 and 2
- a product of two disjoint 2-cycles
- the identity permutation.

(c)
- Let us count the number of different 5-cycles in S_5. We have 5! permutations of 5 elements. Two cycles represent the same group element if and only if they are related by a cyclic permutation. Therefore, there are $\frac{5!}{5} = 24$ different 5-cycles.

- To count the 4 cycles, we have first to choose 4 elements out of 5: there are $\binom{5}{4} = 5$ choices. For each choice, just as above, we have 3! different cycles. Totally we have $5 \cdot 3! = 30$ different 4-cycles in S_5.
- To count the 3 cycles, we have $\binom{5}{3} = \frac{5!}{3!2!} = 10$ choices of 3 numbers, and 2 different 3-cycles for each choice. Totally we have 20 different 3-cycles.
- To count the transpositions, we have to choose 2 numbers out of 5: $\binom{5}{2} = \frac{5!}{2!3!} = 10$.
- The number of products of 3-cycles and 2-cycles is equal to the number of 3-cycles, namely 20.
- To count the products of two transpositions, we have to multiply $\binom{5}{2}$ by $\binom{3}{2}$ and divide by 2, because each transposition will appear in the first choice, and then in the second. We have $\frac{10 \cdot 3}{2} = 15$ such elements.
- There is 1 identity element.

The class equation for 5_5 has the form

$$|S_5| = |\{1\}| + |c_5| + |c_4| + |c_3| + |c_2| + |c_3c_2| + |c_2c_2| = 1 + 24 + 30 + 20 + 10 + 20 + 15 = 120 = 5!$$

Exercise 3. Let G be a group, and $x \in G$ an element. The orbit of the element x under the conjugation action of G on itself, $C_x = \{gxg^{-1}\}_{g\in G}$ is called the conjugacy class of x in G. The subgroup $G_x = \{g \in G : gxg^{-1} = x\}$ is called the centralizer of x in G. Let G be the dihedral group $D_4 = \langle r, s \mid s^2 = 1, r^4 = 1, srs = r^{-1}\rangle$.

(a) For each $x \in D_4$ find its centralizer subgroup.

(b) Describe all the conjugacy classes in $G = D_4$.

(c) Check the Orbit-Stabilizer formula $|C_x| = [G : G_x]$ for each conjugacy class in D_4 and write the class equation for D_4.

Solution 3. (a) Every element in D_4 is of the form r^i or sr^i for some $i = 0, \dots 3$. Let $x = r^i$. We have

$$r^j r^i r^{-j} = r^i, \quad (sr^j)r^i(sr^j)^{-1} = sr^j r^i r^{-j} s = r^{-i}$$

for any $i, j = 0, \dots 3$. Note that $r^i = r^{-i}$ if and only if $i = 2$. Therefore, the centralizer subgroup of r and r^3 is the group of all rotations, $G_r = G_{r^3} = \{1, r, r^2, r^3\}$. The centralizer subgroup of r^2 and of the identity element is the whole group: $G_1 = G_{r^2} = D_4$. For sr^i we have

$$r^j sr^i r^{-j} = sr^{i-2j}, \quad sr^j sr^i (sr^j)^{-1} = sr^j sr^{i-j} s = sr^{2j-i}$$

We have $r^i = r^{i-2j}$ if and only if $j = 0, 2$. Also, $r^i = r^{2j-i}$ if and only if $j - i = 0, j - i = 2$. Therefore we have the following centralizer subgroups for the elements of the form sr^i:

$$\begin{aligned} x = s &\implies G_s = \{1, r^2, s, sr^2\} \\ x = sr &\implies G_{sr} = \{1, r^2, sr, sr^3\} \\ x = sr^2 &\implies G_{sr^2} = \{1, r^2, s, sr^2\} \\ x = sr^3 &\implies G_{sr^3} = \{1, r^2, sr, sr^3\} \end{aligned}$$

(b) Using the computations above we conclude that there are two 1-element conjugacy classes: $\{1\}$, $\{r^2\}$. These elements are central in D_4: they commute with any element in D_4. The remaining rotations form one conjugacy class $C_r = \{r, r^3\}$. Because the conjugation of sr^i result in the elements sr^{i-2j} or sr^{2j-i}, the parity of the power of r is preserved by conjugation. Therefore, the elements of the form sr^i are split between two conjugacy classes $C_s = \{s, sr^2\}$ and $C_{sr} = \{sr, sr^3\}$.

(c) The Orbit-Stabilizer formula has the form:

$$|C_1| = |C_{r^2}| = [D_4 : D_4] = 8 : 8, \quad |C_r| = [D_4 : G_r] = 8 : 4, \quad |C_s| = [D_4 : G_s] = 8 : 4, \quad |C_{sr}| = [D_4 : G_{sr}] = 8 : 4.$$

Let $Z = C_1 \cup C_{r^2} = \{1, r^2\}$ be the center of D_4. The class equation for D_4 is

$$|D_4| = |Z| + |C_r| + |C_s| + |C_{sr}| = 2 + 2 + 2 + 2 = 8.$$

Exercise 4. Consider the dihedral group D_4 acting as the group of symmetries of a square in $\mathbb{R}^2$ centered at the origin. Then each element of D_4 acts on the set of vertices by permutations. This defines an injective (with kernel containing only the identity element) group homomorphism

$$\phi : D_4 \to S_4.$$

Find the subgroup $H = \text{Im}(\phi) \subset S_4$, and write the elements of H as products of disjoint cycles in S_4.

Solution 4. Suppose that $D_4 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$, where r is a rotation counterclockwise by $\pi/2$ and s is the reflection about the axis passing through a pair of vertices. The cycle decompositions of the image of ϕ depend on the numbering of the vertices of the square. For example, if we number the vertices counterclockwise, and let s be the reflection about the axis going through the vertices 1 and 3, we will have the following permutations in the image of ϕ, besides the identity:

$$\phi(r) = (1234),\ \phi(r^2) = (13)(24),\ \phi(r^3) = (1432),\ \phi(s) = (24),\ \phi(sr) = (14)(23),\ \phi(sr^2) = (13),\ \phi(sr^3) = (12)(43).$$

Note that independently of the numbering of the vertices and the choice of ϕ, the image always contains the Klein subgroup $\{1, (12)(34), (13)(24), (14)(23)\}$, where one of the nontrivial elements is a rotation, and the other two - reflections. By the group homomorphism theorem, we have $\mathrm{im}\phi \simeq D_4/\ker\phi \simeq D_4 \subset S_4$. Note that $\mathrm{im}\phi \subset S_4$ is a subgroup of order 8 and index 3.