October 14, 2024

Problem Set 5

Exercise 1. Let G be a group and $H \subset G$ a subgroup of index 2. Show that H is a normal subgroup.

Exercise 2. Let G be a group, and $f : G \to G$ a map defined by $f(g) = g^2$ for any $g \in G$. Find the conditions on G for f to be a group homomorphism.

Exercise 3. Construct an injective (with trivial kernel) group homomorphism from the cyclic group C_4 to the symmetric group S_4. Describe its image in S_4 in terms of the cycle notation. How many different injective homomorphisms from C_4 to S_4 can you define?

Exercise 4. (a) Write the permutations $(2\ 3\ 4\ 5)(4\ 1\ 2)$ and $(3\ 5\ 4)(3\ 6\ 1)(5\ 3)(1\ 2\ 4\ 6)(4\ 3\ 5\ 1)(7\ 3)(1\ 3\ 6)$ as a product of disjoint cycles and then as a product of transpositions.

(b) Determine if the following subsets $H = \{(12)(34); (13)(24); (14)(23); (1)\}$ and $K = \{(12)(24); (13); (34); (1)\}$ are subgroups in $G = S_4$.

(c) What is the order of the element $a = (1\ 3\ 5)(2\ 4\ 6)$ and of the element $b = (1\ 3\ 5)(2\ 5\ 6)$ in S_6? Find an element of order 6 in S_5. *Hint:* Recall that if $ab = ba$ for group elements $a, b \in G$, and the orders $o(a)$ and $o(b)$ are mutually prime, then ab is of order $o(ab) = o(a)o(b)$ (See PS3, Ex. 3(c)).

Exercise 5. It is known that the symmetric group S_n is generated by all transpositions $\{(ik)\}_{1\le i<k\le n}$. Show that S_n is also generated by the following sets of elements:

(a) All transpositions of the form $\{(1, i)\}$, $2 \le i \le n$.

(b) The transposition (12) and the n-cycle $(123\ldots n)$.

Hint: For any $\pi, \rho \in S_n$, the cycle decomposition of $\pi\rho\pi^{-1}$ is obtained by replacing each integer i in the cycle decomposition of ρ with the integer $\pi(i)$.

Exercise 6. Let G be a group and H a subgroup in G. We say that H is *proper* in G if H is not equal to G, and *maximal proper* in G if H is not equal to G and no other proper subgroup of G contains H.

(a) Let H be a subgroup of $(\mathbb{Z}, +)$. Show that H is maximal proper if and only if $H = p\mathbb{Z}$ for a prime number p.

(b) Find all subgroups in $(\mathbb{Z}, +)$ that contain $72\mathbb{Z}$ as a proper subgroup. Which of them are maximal proper subgroups?