September 30, 2024

Problem Set 3

Exercise 1. Determine which of the following groups are cyclic.

(a) $(\mathbb{Z}/12\mathbb{Z})^*, \ \cdot)$

(b) $(\mathbb{Z}/12\mathbb{Z}, +)$

(c) $((\mathbb{Z}/8\mathbb{Z})^*, \ \cdot)$

Exercise 2. For each of the groups below, find the order of the element $g \in G$;

(a) $G = ((\mathbb{Z}/20\mathbb{Z})^*, \ \cdot)$, $g = [3]_{20}$.

(b) $G = ((\mathbb{Z}/24\mathbb{Z})^*, \cdot)$, $g = [5]_{24}$ and $g = [11]_{24}$.

(c) $G = \mathrm{GL}_2(\mathbb{R})$, $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

Exercise 3. (a) Find the last digit of 7^{1000}.

(b) Show that 72 divides $53^{48} - 1$.

(c) Show that the number $a = (29^{16} + 28^{16})(29^8 + 28^8)(29^4 + 28^4)(29^2 + 28^2)(29 + 28)$ is divisible by 51. Hint: Use Euler's theorem.

Exercise 4. (a) Show that $a^{13} \equiv a \pmod{2730}$ for any integer a.

(b) Let q and p be two distinct primes. Show that $p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}$.

(c) Let p be a prime different from 2 and 5. Show that p divides an infinite number of elements of the sequence $9, 99, 999, 9999, \ldots$. (Hint: note that each element of the sequence can be written as $10^a - 1$ for an integer a.)

Exercise 5. Let C_n denote the cyclic group of order $n \in \mathbb{Z}^+$.

(a) Describe all group homomorphisms $C_n \to C_n$. How many are there?

(b) The kernel of a group homomorphism $C_n \to C_n$ is the set of the elements of C_n that are mapped to 1. A homomorphism from a group to itself is an automorphism if its kernel is trivial (equal to $\{1\}$). Describe all group automorphisms $C_n \to C_n$. How many are there?

(c) Describe all group homomorphisms $C_n \to C_m$ for $m, n \in \mathbb{Z}^+$, $m \neq n$. How many are there?