September 9, 2024

Problem Set 1

Exercise 1. *Use the fundamental theorem of arithmetic and the well-ordering principle to show that for a prime number p, the square root $\sqrt{p}$ is irrational.*

Exercise 2. *Show that the strong induction principle implies the well-ordering principle.*
Strong induction principle: Let $P(n)$ be a statement that depends on $n \in \mathbb{N}$. If

1. *$P(0)$ is true, and*
2. *$\{P(0), P(1), \ldots P(n)\}$ imply $P(n+1)$ for any $n \in \mathbb{N}$,*

then $P(n)$ is true for all $n \in \mathbb{N}$.

Well-ordering principle: Every nonempty subset $S \subset \mathbb{N}$ contains a least element.

Exercise 3. *Use the Euclidean algorithm to find the greatest common divisor* $\gcd(a, b)$ *for the following integers:*

(a) $a = 73$ *and* $b = 12$.

(b) $a = 101$ *and* $b = -32$.

(c) $a = 9050$ *and* $b = 1004$.

In each case find integers $x, y \in \mathbb{Z}$ such that $xa + yb = \gcd(a, b)$.

Exercise 4.

1. *Show that if $a, b \in \mathbb{Z}^*$ and $d = \gcd(a, b)$, then the equation*

$$ax + by = c$$

has a solution in integer numbers if and only if $c \in d\mathbb{Z}$.

2. *Suppose that $a, b \in \mathbb{Z}^*$ and $c \in \mathbb{Z}$ are such that the equation $ax+by = c$ has a solution (x_0, y_0) in integer numbers. Find all possible pairs of integer solutions (x, y) in terms of x_0, y_0, a, b.*

Exercise 5. *Bézout's theorem states that two integers s and t are coprime if and only if there exist two integers x and y such that $xs + yt = 1$. Use Bézout's theorem to show that if an integer n divides a product of two integers a and b, and n is coprime with a, then n divides b.*