November 18, 2024

**Graded assignment**
**Due on Moodle by 20:00 November 25, 2024**

---

Let $n = \text{day} \in \{1, 2, 3, \dots, 31\}$ be the date of your birth[1], and let $m \in \mathbb{N}$ be the last two digits of your SCIPER number, not counting the zeros.[2] Please write the integers $n$ and $m$ clearly on top of your solution. These numbers will be used in exercises below.

Suppose for this solution that n =10, m= 46.

**Exercise 1.** (7pts) Let $n$ and $m$ be defined as above. Consider the numbers $m^2$ and $nm$ as strings of digits $(a_1a_2...)$ and $(b_1b_2...)$. Let $a = (a_1, a_2, ...)$ and $b = (b_1, b_2, ....)$ be the elements in $S_9$, where you discard all zeros and repeated digits[3].

(a) Let $x = a \cdot b \cdot (132) \in S_9$. Write $x$ as a product of disjoint cycles.

(b) Find the order of $x$ in $S_9$. Justify your answer.

(c) Let $H = \langle x \rangle \subset S_9$ be the subgroup generated by $x$ in $S_9$. Find the number of elements in the orbit of 1 under the action of $H$. Find the stabilizer subgroup of 1 in $H$ and check that the orbit-stabilizer theorem holds for the orbit of 1 under the action of $H$.

(d) Describe the elements in the conjugacy class of $x$ in $S_9$ and find the number of these elements.

(e) We have $H = \langle x \rangle \subset S_9$ is a subgroup. What is the smallest $k$ such that the group $S_k$ has a subgroup isomorphic to $H$?

**Solution 1.** (a) $m^2 = 46^2 = 2116$, $nm = 460$, and we have $a = (216)$, $b = (46)$. Then

$$x = (216)(46)(132) = (13)(264).$$

(b) The order of $x$ is 6 since $x = (13)(264)$ is a product of a 2-cycle and a 3-cycle.

(c) $H = \langle (13)(264) \rangle = \{1, (13)(264), (246), (13), (264), (13)(246), \}$. We have $|H| = 6$. The orbit of 1 by the action of $H$ is $O_1 = \{1, 3\}$, so that $|O_1| = 2$. The stabilizer subgroup of 1 in $H$ is $\{1, (246), (264)\}$. The orbit-stabilizer theorem holds: $|H|/|Stab_1| = |O_1|$, where $|H| = 6$, $|Stab_1| = 3$ and $|O_1| = 2$.

(d) The conjugacy class of $x = (13)(246) \in S_9$ is determined by the cycle type of the element, and contains all elements of $S_9$ that are disjoint products of a 2-cycle and a 3-cycle. The number of such elements is

$$\binom{9}{3} \cdot 2 \cdot \binom{6}{2} = \frac{9 \cdot 8 \cdot 7 \cdot 2}{3 \cdot 2} \cdot \frac{6 \cdot 5}{2} = 56 \cdot 3 \cdot 15 = 2520.$$

We multiplied by 2 because for each choice of three elements there are two distinct 3-cycles.

(e) $x = (13)(264)$ and it generates a subgroup $H \subset S_9$ of order 6, which is cyclic (since the order of $x$ is 6). For a symmetric group $S_k$ to have a subgroup isomorphic to $H$ is equivalent for it to have an element of order 6. An element of a symmetric group of order 6 is either a 6-cycle or a product of disjoint 2-cycle and 3-cycle, because the order of a disjoint product of cycles equals to the lcm of their lengths. Therefore, the smallest symmetric group that contains a subgroup isomorphic to $H$ is $S_5$, and the subgroup is generated for example by the element $(12)(345)$.

---

[1] example: birthday 16.08 gives n = 16

[2] example: SCIPER 123209 gives $m = 29$, SCIPER 123450 gives $m = 45$, SCIPER 123400 gives $m = 34$.

[3] $n = 16$, $m = 29$ gives $m^2 = 841$, $nm = 464$ and $a = (841)$ and $b = (46)$

**Exercise 2.** (8pts) Let $n$ and $m$ be defined as above. Let $C_n$ and $C_m$, $C_{nm}$, $C_{n^2}$ and $C_{m^2}$ be the cyclic groups of orders $n$, $m$, $nm$, $n^2$, $m^2$ respectively.

(a) For the given $n$ and $m$ classify the abelian groups of order $nm$. List the elementary divisors and invariant factors for each group.

(b) For the given $n$ and $m$ consider the groups

$$G_1 = C_n \times C_m \times C_n \times C_m, \qquad G_2 = C_{n^2} \times C_{m^2}, \qquad G_3 = C_{nm} \times C_{nm}.$$

Find the biggest order of an element in each group.

(c) Are any of the groups $G_1, G_2, G_3$ isomorphic for the given $n$ and $m$? Justify your answer.

(d) More generally, let $n, m > 1$ be arbitrary natural numbers. Find the necessary and sufficient conditions for the groups $C_{n^2} \times C_{m^2}$ and $C_{nm} \times C_{nm}$ to be isomorphic.

(e) Similarly, let $n, m > 1$ be arbitrary natural numbers. Find the necessary and sufficient conditions for the groups $C_{nm} \times C_{nm}$ and $C_n \times C_m \times C_n \times C_m$ to be isomorphic.

**Solution 2.** (a) $|G| = 460 = 2^2 \cdot 5 \cdot 23$. Based on the classification theorem for finite abelian group, we know that an abelian group $|G| = n$ is isomorphic to a direct product of cyclic groups $C_{p_1^{a_1}} \times \dots C_{p_r^{a_r}}$ of powers of primes in the prime factorization of $n = p_1^{a_1} \dots p_r^{a_r}$. The partitions of 2 are $(2), (1,1)$, other primes occur multiplicity-free. Therefore there are exactly two non-isomorphic groups of order 460, namely $C_4 \times C_5 \times C_{23}$ and $C_2 \times C_2 \times C_5 \times C_{23}$. The elementary divisors of the groups are $(4, 5, 23)$ and $(2, 2, 5, 23)$. The invariant factors of the groups are $(460)$ and $(230, 2)$.

(b) The maximal order of an element in $G$ is the order of the maximal cyclic subgroup in $G$. A direct product of two cyclic groups is cyclic if and only if the orders of the groups are coprime. Therefore, the cyclic subgroup of maximal order is the least common multiple of of the elementary divisors of the group. Equivalently, this is the highest invariant factor of the group. In the given case we will write the groups in terms of their elementary divisors and invariant factors to find the highest invariant factor for each group:

$$G_1 = C_{10} \times C_{46} \simeq C_2 \times C_2 \times C_5 \times C_{23} \simeq C_{230} \times C_2.$$

$$G_2 = C_{10} \times C_{46} \times C_{10} \times C_{46} \simeq C_2 \times C_2 \times C_2 \times C_2 \times C_5 \times C_5 \times C_{23} \times C_{23} \simeq C_{230} \times C_{230} \times C_2 \times C_2.$$

$$G_3 = C_{100} \times C_{46^2} \simeq C_4 \times C_{25} \times C_4 \times C_{23^2} \simeq C_4 \times C_4 \times C_{25} \times C_{23^2} \simeq C_{23^2 \cdot 25 \cdot 4} \times C_4.$$

$$G_4 = C_{460} \times C_{460} \simeq C_4 \times C_5 \times C_{23} \times C_4 \times C_5 \times C_{23} \simeq C_4 \times C_4 \times C_5 \times C_5 \times C_{23} \times C_{23} \simeq C_{460} \times C_{460}.$$

Therefore, the biggest order of an element is 230 in $G_1$ and in $G_2$, $23^2 \cdot 25 \cdot 4$ in $G_3$ and 460 in $G_4$.

(c) Since the presentation in terms of the elementary divisors and invariant factors determines the group uniquely, we can conclude that none of the groups $G_2, G_3, G_4$ are isomorphic.

(d) Clearly if $n = m$ then the groups $C_{n^2} \times C_{m^2}$ and $C_{nm} \times C_{nm}$ are isomorphic. Suppose $n \neq m$. Suppose first that there exists a prime $p$ which enters the prime factorization of $n$ but not of $m$. Then $n = p^a t$, where $\gcd(t, p) = 1$ and $\gcd(p, m) = 1$. Then $C_{n^2} \times C_{m^2}$ has the form $C_{p^{2a}} \times G'$, with $|G'|$ not divisible by $p$, whereas $C_{nm} \times C_{nm}$ has a direct factor $C_{p^a} \times C_{p^a} \times G''$, where $|G''|$ is not divisible by $p$. The factors $C_{p^{2a}}$ and $C_{p^a} \times C_{p^a}$ are the only parts containing powers of the prime $p$ that appear in the elementary divisor- decomposition of the groups $C_{n^2} \times C_{m^2}$ and $C_{nm} \times C_{nm}$. The elementary divisors determine the group uniquely. Since $C_{p^{2a}} \not\simeq C_{p^a} \times C_{p^a}$, the groups are not isomorphic.
Now suppose that $n$ and $m$ have prime factorization with the same set of primes, but $n = p^a t$ and $m = p^b d$, where $\gcd(p, t) = 1$ and $\gcd(p, d) = 1$. Then $C_{n^2} \times C_{m^2} \simeq C_{p^{2a}} \times C_{p^{2b}} \times G'$, where $|G'|$ is not divisible by $p$ and $C_{mn} \times C_{mn} \simeq C_{p^{a+b}} \times C_{p^{a+b}} \times G''$, where $|G''|$ is not divisible by $p$. Since distinct abelian groups of order $p^k$ are in bijection with the partitions of $k$, we have that $C_{p^{2a}} \times C_{p^{2b}} \simeq C_{p^{a+b}} \times C_{p^{a+b}}$ if and only if $2a = a + b = 2b$, we conclude that $a = b$ and finally $n = m$. Finally, the groups $C_{n^2} \times C_{m^2}$ and $C_{nm} \times C_{nm}$ are isomorphic if and only if $n = m$.

(e) If $\gcd(n, m) = 1$, then $C_{nm} \simeq C_n \times C_m$, and therefore the groups $C_{nm} \times C_{nm}$ and $C_n \times C_m \times C_n \times C_m$ are isomorphic. Now suppose that $\gcd(n, m) \neq 1$ and there exists a prime $p$ such that $p^a$ is the highest power of $p$ in $n$ and $p^b$ the highest power of $p$ in $m$. Then $C_{p^{a+b}} \times C_{p^{a+b}}$ is a direct factor in the group $C_{nm} \times C_{nm}$, while $C_{p^a} \times C_{p^b} \times C_{p^a} \times C_{p^b}$ is a direct factor of the group $C_n \times C_m \times C_n \times C_m$. These groups can be isomorphic if and only if either $a = 0$ or $b = 0$, which means that $p$ divides at most one of the numbers $n$, $m$. Therefore, the groups $C_{nm} \times C_{nm}$ and $C_n \times C_m \times C_n \times C_m$ are isomorphic if and only if $\gcd(n, m) = 1$.