Algebra MATH-310

Lecture 10

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Plan of the course

1. Integers: 1 lecture
2. Groups: 6 lectures
3. Rings and fields: 5 lectures
4. Review: 1 lecture

Today: Rings: lecture 3

(a) Chinese remainder theorem.

(b) CRT for integers.

(c) Degree of a polynomial.

(d) Euclidean division for polynomials.

(e) Euclidean domains.

Recall: ring homomorphism and the direct product of rings

Definition

A map $f : A \to B$ is a ring homomorphism if

- $f(a+b) = f(a) + f(b)$,
- $f(a \cdot b) = f(a) \cdot f(b)$,
- $f(1_A) = 1_B$.

A bijective ring homomorphism is a ring isomorphism.

Definition

A direct product of two rings A, B is defined as the set of pairs

$$A \times B = \{(a, b),\ a \in A, b \in B\}$$

with component-wise operations and the neutral elements $(0_A, 0_B)$, $(1_A, 1_B)$.

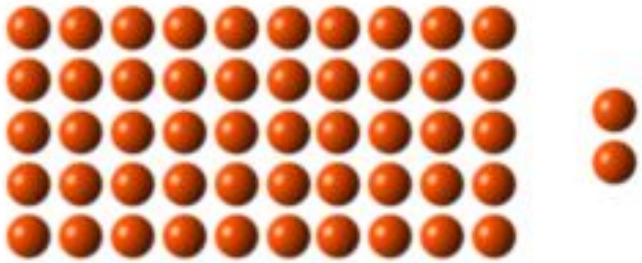
Today: the Chinese remainder theorem

Sunzi Suanjing

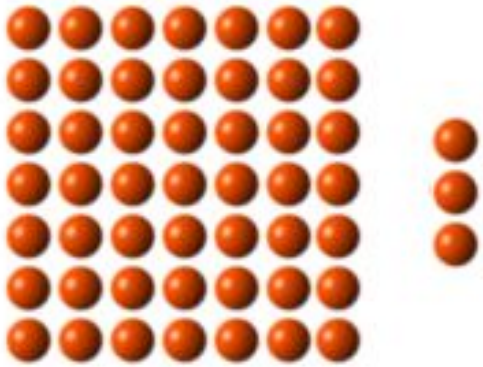
3rd - 5th century CE.

If a collection of balls are arranged in rows of 3, there is one ball left over

If arranged in rows of 5, there are two balls left over



If arranged in rows of 7, there are three balls left over



What is the minimal number of balls? See the solution below.

Chinese remainder theorem

Theorem

Let A be a commutative ring, $I, J \subset A$ two ideals such that $I + J = A$. Then there is a ring isomorphism

$$f : A/(I \cap J) \simeq A/I \times A/J$$

$$f([x]_{I\cap J}) = ([x]_I, [x]_J).$$

Proof:

(1) f is a ring homomorphism:

$$f: \underset{\in A}{x} \longrightarrow ([x]_I, [x]_J)$$

$$1 \longrightarrow ([1]_I, [1]_J)$$

$$0 \longrightarrow ([0]_I, [0]_J)$$

and the ring operations are respected.

Chinese remainder theorem

(2) Surjectivity. Show that $\forall a_1, a_2 \in A \ \exists a \in A$ s.t. $a \equiv a_1 \pmod I$, $a \equiv a_2 \pmod J$

Since $I + J = A \Rightarrow \underbrace{a_1 - a_2}_{\in A} = \underbrace{-i}_{\in I} + \underbrace{j}_{\in J} \Leftrightarrow a_1 + i = a_2 + j \overset{\text{def}}{=} a \in A$

$\Rightarrow a \equiv a_1 \pmod I \quad a \equiv a_2 \pmod J$

$\Rightarrow f : x \to ([x]_I, [x]_J)$ is surjective.

(3) Injectivity. Suppose $b \in A : b \equiv a_1 \pmod I$, $b \equiv a_2 \pmod J$

$\Rightarrow b = a_1 + \overset{\in I}{i'} = a_2 + \overset{\in J}{j'} \Rightarrow a - b = \underbrace{i - i'}_{\in I} = \underbrace{j - j'}_{\in J}$

$\Rightarrow a - b \in I \cap J$

$\Rightarrow f : A/_{I \cap J} \to A/_I \times A/_J$ is injective.

$\Rightarrow f : A/_{I \cap J} \to A/_I \times A/_J$ is a rng isomorphism.

The case $A = \mathbb{Z}$.

Corollary

Let $n, m \in \mathbb{Z}$ be coprime: $\gcd(n, m) = 1$. Then for any numbers $a_1, a_2 \in \mathbb{Z}$ there exist $a \in \mathbb{Z}$ such that

$$\begin{cases} a \equiv a_1 \pmod{n} \\ a \equiv a_2 \pmod{m} \end{cases}$$

The set of solutions of this pair of congruences is $\{a + (nm)\mathbb{Z}\}$.

Proof: $\gcd(n,m) = 1 \Rightarrow \exists x, y \in \mathbb{Z} : xn + ym = 1 \Rightarrow (n) + (m) = \mathbb{Z}$

$\Rightarrow$ By CRT $\mathbb{Z}/(n\mathbb{Z} \cap m\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ ($n\mathbb{Z}$, $m\mathbb{Z}$) $\Rightarrow$ for any pair $(a_1 \bmod n)$, $(a_2 \bmod m)$

there exists $a \in \mathbb{Z} : a \equiv a_1 \pmod{n}$, $a \equiv a_2 \pmod{m}$, a is unique up to the ideal $(n\mathbb{Z}) \cap (m\mathbb{Z}) = (nm)\mathbb{Z}$

$\Rightarrow$ the solutions are $\{a + nm\mathbb{Z}\}$ ($\gcd(n,m) = 1$)

Generalization to $n > 2$ for $A = \mathbb{Z}$

Theorem

Let $d_1, d_2, \ldots d_r \in \mathbb{Z}$ be pairwise coprime, i.e. $\gcd(d_i, d_j) = 1$ for any $i \neq j$. Then for any numbers $a_1, a_2, \ldots a_r \in \mathbb{Z}$ there exist $a \in \mathbb{Z}$ such that

$$\begin{cases} a \equiv a_1 \pmod{d_1} \\ a \equiv a_2 \pmod{d_2} \\ \quad \ldots. \\ a \equiv a_r \pmod{d_r} \end{cases}$$

The number $a \in \mathbb{Z}$ is unique up to the ideal $(d_1 d_2 \ldots d_r) \subset \mathbb{Z}$. The set of solutions is given by $\{a + (d_1 d_2 \ldots d_r)\mathbb{Z}\}$.

Proof: by induction on r. See rings.pdf

Example.

Find $a \in \mathbb{Z}$ such that

$$\begin{cases} a \equiv 1 \pmod 3 \\ a \equiv 2 \pmod 5 \\ a \equiv 3 \pmod 7 \end{cases}$$

See rings.pdf for an algorithm to solve systems of congruences.

1. Explain why there is a solution. By CRT, since 3, 5, 7 are pairwise coprime
2. Describe the set of all integer solutions.
3. Find the smallest positive solution.

$a = 3k + 1 = 5t + 2$, for example $a = 7$

$\begin{cases} a \equiv 7 \pmod{15} \\ a \equiv 3 \pmod 7 \end{cases}$ $\quad a = 15m + 7 = 7n + 3 \quad \Rightarrow a = 52.$

$7n - 15m = 4 \qquad m = 3, n = 7$

$49 - 45 = 4$

$\Rightarrow a = \{52 + 105\,\mathbb{Z}\}$ all solutions $\qquad 105 = 3 \cdot 5 \cdot 7.$

the smallest positive solution is 52.

Application: multiplicativity of the totient function $\varphi(n)$

Definition

Let A be a commutative ring. Then its invertible elements with respect to the multiplication form a group that is called the group of units and denoted A^*.

Example: $A = \mathbb{Z}$. $\Rightarrow$ $\mathbb{Z}^* = \{\pm 1\}$

Remark: If $A \simeq B$ are isomorphic rings, then their groups of units are also isomorphic: $A^* \simeq B^*$.

Application: multiplicativity of the totient function $\varphi(n)$

Theorem

Let $n, m \in \mathbb{Z}$ such that $\gcd(n, m) = 1$. Then $\varphi(nm) = \varphi(n)\varphi(m)$.

Proof: By CRT $\mathbb{Z}/_{nm\mathbb{Z}} \simeq \mathbb{Z}/_{n\mathbb{Z}} \times \mathbb{Z}/_{m\mathbb{Z}}$ =>

$\left(\mathbb{Z}/_{nm\mathbb{Z}}\right)^* \simeq \left(\mathbb{Z}/_{n\mathbb{Z}} \times \mathbb{Z}/_{m\mathbb{Z}}\right)^* \simeq \left(\mathbb{Z}/_{n\mathbb{Z}}\right)^* \times \left(\mathbb{Z}/_{m\mathbb{Z}}\right)^*$ direct product of groups of units

$\left|\left(\mathbb{Z}/_{nm\mathbb{Z}}\right)^*\right| = \varphi(nm)$, $\left|\left(\mathbb{Z}/_{n\mathbb{Z}}\right)^*\right| = \varphi(n)$, $\left|\left(\mathbb{Z}/_{m\mathbb{Z}}\right)^*\right| = \varphi(m)$

$\Rightarrow \varphi(mn) = \varphi(n)\varphi(m)$ ▨

prime factorization: $\{p_i\}$ are distinct primes.

This can be used to compute $\varphi(p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}) = \varphi(p_1^{a_1}) \cdot \dots \varphi(p_k^{a_k})$. =

$= \left(p_1^{a_1} - p_1^{a_1-1}\right)\left(p_2^{a_2} - p_2^{a_2-1}\right) \dots \left(p_k^{a_k} - p_k^{a_k-1}\right)$

Converse to the Chinese remainder theorem

Theorem

$$\mathbb{Z}/(nm)\mathbb{Z} \simeq \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \iff \gcd(n, m) = 1.$$

Proof: $\gcd(n,m)=1 \Rightarrow$ by CRT $\mathbb{Z}/_{nm\mathbb{Z}} \simeq \mathbb{Z}/_{n\mathbb{Z}} \times \mathbb{Z}/_{m\mathbb{Z}}$

If $\mathbb{Z}/_{nm\mathbb{Z}} \simeq \mathbb{Z}/_{n\mathbb{Z}} \times \mathbb{Z}/_{m\mathbb{Z}} \Rightarrow \tau\left(\mathbb{Z}/_{mn\mathbb{Z}}\right) = \tau\left(\mathbb{Z}/_{n\mathbb{Z}} \times \mathbb{Z}/_{m\mathbb{Z}}\right) \Rightarrow nm = \operatorname{lcm}(n,m)$

$= nm$ $=$ $= \operatorname{lcm}(n,m)$ $\Leftrightarrow \gcd(n,m)=1$

Examples:

$\gcd(5,16)=1$

$\mathbb{Z}/80\mathbb{Z} \approx \mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$

$\mathbb{Z}/80\mathbb{Z} \not\simeq \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$

$\gcd(8,10)=2$

$\gcd(6,9)=3$ $\qquad$ $\gcd(2,27)=1$

$\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} \not\simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/27\mathbb{Z} \simeq \mathbb{Z}/_{54\mathbb{Z}}$

$\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$

$\gcd(2,3)=1$

Poll:

Let $n, m \in \mathbb{N}$, $n, m \geq 2$ and p is a prime such that $\gcd(n, m) = 1$, $\gcd(n, p) = 1$ and $\gcd(m, p) = 1$. Then

A: $\varphi(p^2nm) = \varphi(pn)\varphi(pm)$

B: $\varphi(p^2nm) > \varphi(pn)\varphi(pm)$

C: $\varphi(p^2nm) < \varphi(pn)\varphi(pm)$

D: The relation between $\varphi(p^2nm)$ and $\varphi(pn)\varphi(pm)$ depends on the numbers n, m, p.

$\varphi(p^2nm) = \varphi(p^2)\varphi(n)\varphi(m) = (p^2-p)\varphi(n)\varphi(m) > \varphi(pn)\varphi(pm) = \varphi(p)\varphi(n)\varphi(p)\varphi(m) = (p-1)^2\varphi(n)\varphi(m)$

$p^2 - p = p(p-1) > (p-1)^2$

Conclusions:

- If A is a commutative ring and $I, J \subset A$ two ideals such that $I + J = A$, then
$$A/I \times A/J \simeq A/(I \cap J).$$
- $\mathbb{Z}/(nm)\mathbb{Z} \simeq \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \iff \gcd(n, m) = 1.$
- $\varphi(nm) = \varphi(n)\varphi(m) \iff \gcd(n, m) = 1.$

Next goal: CRT for polynomial rings

Definition

Let A be a commutative ring. Then

$$A[x] = \{a_0 + a_1 x + \ldots + a_n x^n\}_{n \in \mathbb{N}}, \quad a_0, a_1, \ldots \in A$$

is the ring of polynomials with coefficients in A with respect to the usual addition and multiplication of polynomials. We have $0 \in A[x]$ and $1 \in A[x]$ same as in A.

Definition

If $f(x) \in A[x]$ is nonzero, define the degree of

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n$$

as the largest $n \in \mathbb{N}$ such that $a_n \neq 0$. Notation: $\deg(f) = n$.
If $f(x) = 0$, we set $\deg(f) = -\infty$. If $f(x) = a_0 \neq 0$, then $\deg(f) = 0$.

Polynomials with coefficients in an integral domain

Proposition

Let A be an integral domain. Then we have

1. $\deg(f+g) \leq \max(\deg(f), \deg(g))$.
2. $\deg(f \cdot g) = \deg(f) + \deg(g)$.

Proof:

(1) $\deg(f+g) = \max(\deg f, \deg g)$ unless $\deg f = \deg g$ and $a_n = -b_n$

Ex: $(3x^5 + 2x^2 + 3x - 1) + (-3x^5 - 7x^4 + 2x^2 - 2) = -7x^4 + 4x^2 + 3x - 3$

$\deg = 5$ $\deg = 5$ $\deg = 4$

(2) $f(x) \cdot g(x) = (a_0 + a_1 x + \dots a_n x^n)(b_0 + b_1 x + \dots + b_m x^m) = a_n b_m x^{n+m} + \text{lower terms}$

$\neq 0$ $\neq 0$ $\neq 0$ A is an integral domain.

$\Rightarrow \deg(f \cdot g) = \deg f + \deg g$; If $f(x) = 0 \Rightarrow \deg f = -\infty$

$\forall g(x)$ $f(x) \cdot g(x) = 0 \Rightarrow \deg f \cdot g = -\infty + m = -\infty$

$= \deg g$

Polynomials with coefficients in an integral domain

Proposition

Let A be an integral domain. Then

1. *$A[x]$ is an integral domain.*
2. *The units (invertible elements) of $A[x]$ are the units of A.*

Proof:

(1) $f(x)\cdot g(x) = 0 \iff \deg(f\cdot g) = -\infty = \deg f + \deg g \iff \begin{cases} \deg f = -\infty \\ \deg g = -\infty \end{cases}$
$\iff f(x) = 0$ or $g(x) = 0$.

(2) $f(x)\cdot g(x) = 1 \iff \deg(f\cdot g) = 0 = \deg f + \deg g \Rightarrow \deg f = \deg g = 0$.
$\Rightarrow f(x) = a_0$, $g(x) = b_0$ s.t. $a_0 b_0 = 1 \in A$
$\Rightarrow a_0, b_0$ are units in A.

Examples: $\mathbb{R}[x]$, $\mathbb{Q}[x]$, $\mathbb{Z}[x]$ integral domains; $\mathbb{Z}/6\mathbb{Z}[x]$ is not: $2x\cdot 3x = 0$

Euclidean division for polynomials in $\mathbb{F}[x]$

Theorem

Let $\mathbb{F}$ be a field. Let $f(x), d(x) \in \mathbb{F}[x]$ such that $\deg(d) \geq 1$. Then there exist polynomials $d(x), r(x) \in \mathbb{F}[x]$ such that

$$f(x) = q(x)d(x) + r(x),$$

and either $r(x) = 0$, or $\deg(r) < \deg(d)$.

Proof: If $\deg f < \deg d \Rightarrow f(x) = 0 \cdot d(x) + f(x) \Rightarrow f(x) = r(x)$

If $\deg f \geq \deg d \Rightarrow f(x) = a_0 + \dots + a_m x^m$, $d(x) = b_0 + \dots + b_n x^n$, $n \leq m$

$\Rightarrow f(x) - d(x) \cdot \frac{a_m}{b_n} x^{m-n} = p_1(x)$, $\deg p_1 < \deg f$.

If $\deg p_1 > \deg d$, repeat $\Rightarrow f(x) - d(x)\frac{a_m}{b_n} x^{m-n} - d x \frac{a_{m-1}}{b_n} x^{m-n-1} - \dots$

until $\deg\left(f - d(x)\, q(x)\right) < \deg d \Rightarrow f(x) = d(x)q(x) + r(x)$

$\deg r < \deg d$.

Euclidean division for polynomials in $\mathbb{F}[x]$

The degree is strictly decreasing $\Rightarrow$ the process terminates.

Example: $f(x) = 3x^5 + x^3 - 2x^2 + 1, \quad d(x) = x^2 - 2 \in \mathbb{R}[x]$.

$$
\begin{array}{r|l}
3x^5 + x^3 - 2x^2 + 1 & x^2 - 2 \\
\cline{2-2}
-\,(3x^5 - 6x^3) & 3x^3 + 7x - 2 \\
\cline{1-1}
7x^3 - 2x^2 + 1 & \\
-\,(7x^3 - 14x) & \\
\cline{1-1}
-2x^2 + 14x + 1 & \\
-\,(-2x^2 + 4) & \\
\cline{1-1}
14x - 3 &
\end{array}
$$

$\Rightarrow q(x) = 3x^3 + 7x - 2$

$r(x) = 14x - 3$

$1 = \deg r < \deg d = 2$

Euclidean domains

Definition

A commutative ring A is a Euclidean domain if

1. A is an integral domain,
2. There exist a function $\nu : A \setminus \{0\} \to \mathbb{N}$ such that for all $a, b \in A$, $b \neq 0$, there exist $q, r \in A$ such that $a = qb + r$ and either $r = 0$, or $\nu(r) < \nu(b)$.

Examples:

1. $\mathbb{Z}$ with $\nu(n) = |n| \in \mathbb{N}$. $a = bq + r$, $|r| < |b|$
2. Any field with $\nu : \mathbb{F} \setminus \{0\} \to \mathbb{N}$ any function $a = bq + 0$, $r = 0$
3. $\mathbb{F}[x]$, $\mathbb{F}$ a field with $\nu(f(x)) = \deg(f)$. $f(x) = q(x) \cdot d(x) + r(x)$; either $r(x) = 0$ or $\deg r < \deg d$.
4. * $\mathbb{Z}[i] = \{a + bi\}_{a,b \in \mathbb{Z}}$ with $\nu(a + ib) = a^2 + b^2$. (not a part of the course).
   Gaussian integers

Euclidean domains

Proposition

A Euclidean domain is a PID (principal ideal domain).

Proof: Let E be a Euclidean domain, $I \subset E$ an ideal.

If $I = \{0\}$, then $I = (0)$, done.

If $I \neq \{0\}$ let $d \in I$, $d \neq 0$ s.t. $\nu(d)$ is the minimum on I.

Suppose $a \in I \Rightarrow \exists\, q, r$: $a = qd + r \Rightarrow r \in I \Rightarrow$ ($a \in I$, $qd \in I$)

$$\begin{cases} \nu(r) < \nu(d) \\ r = 0. \end{cases}$$

$\nu(r) < \nu(d)$ is impossible since $\nu(d)$ is minimal in I

$\Rightarrow r = 0 \Rightarrow a = qd \quad \forall a \in I$

$\Rightarrow I = (d)$ is principal $\Rightarrow E$ is a PID.

(E is an integral domain by def of Euclidean domain) ∎

Conclusions

Let $\mathbb{F}$ be a field. Then the ring $\mathbb{F}[x]$ is a PID, meaning that any ideal in $\mathbb{F}[x]$ is generated by a single element.

$\forall$ nonzero elt is invertible — admit euclidean division — ideals gen by 1 elt — no nontrivial zero divisors — multiplication is commutativ

Fields $\subset$ Euclidean domains $\subset$ PID $\subset$ Integral domains $\subset$ Comm. rings

$\mathbb{R}, \mathbb{C}$ — $\mathbb{Z}$, $\mathbb{R}[x]$ — $\mathbb{Z}$, $\mathbb{C}[x]$ — $\mathbb{R}[x,y]$ — $\mathbb{Z}/6\mathbb{Z}$

$\mathbb{Z}/p\mathbb{Z}$, p prime

$\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$

PID, not Euclidean