Exercise Sheet 9

Introduction to Partial Differential Equations (W. S. 2024/25)

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- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

Exercise 1. Prove that every function $u \in W^{1,p}(\Omega)$, with $\Omega = (a,b)$ a bounded interval, and $1 \leq p \leq \infty$ has a unique continuous representative,[1] which is moreover Hölder continuous $C^{0,\alpha}([a,b])$ for any $\alpha \in \left[0, 1 - \frac{1}{p}\right]$.

As a corollary of this claim, prove that

- $W^{1,p}(a,b) \hookrightarrow L^q(a,b), \quad$ for all $q \in [1, \infty]$,
- $W^{1,p}(a,b) \hookrightarrow C^{0,\alpha}([a,b]), \quad$ for all $\alpha \in \left[0, 1 - \frac{1}{p}\right]$.

Hint: let $y \in (a,b)$ and define

$$\tilde{u}(x) = \int_y^x u'(t) \, \mathrm{d}t, \quad x \in [a,b].$$

Use Lebesgue's dominated convergence theorem to show that $\tilde{u} \in C^0([a,b])$. Prove that $\tilde{u}$ is a continuous representative of u.

Next, for the Hölder continuity, we prove that

$$\|u\|_{C^0([a,b])} \leq C\|u\|_{W^{1,p}}$$
$$\|u\|_{C^{0,1-1/p}([a,b])} = \|u\|_{C^0([a,b])} + \sup_{x_1,x_2 \in (a,b)} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^{1-1/p}} \leq C\|u\|_{W^{1,p}},$$

which yields $u \in C^{0,1-1/p}([a,b])$ and, more generally, $u \in C^{0,\alpha}([a,b])$ for any $\alpha \in [0, 1-1/p]$.

To prove these inequalities, (1) write $|u(x)| = \left| \frac{1}{b-a} \int_a^b u(x) dy \right|$ and estimate appropriately; (2) write $|u(x_1) - u(x_2)| = |\tilde{u}(x_1) - \tilde{u}(x_2)|$ and estimate appropriately.

Solution: Let $y \in (a,b)$ and define

$$\tilde{u}(x) = \int_y^x u'(t) dt, \quad x \in [a,b].$$

[1] We recall that a function $u \in W^{1,p}(\Omega)$ admits a *continuous representative* in $\bar{\Omega}$ if there exists a function $\hat{u} \in C^0(\bar{\Omega})$ such that $u = \hat{u}$ almost everywhere in Ω (i.e. $\hat{u}$ belongs to the equivalence class of u).

One easily checks that $\tilde{u} \in C^0([a,b])$ (using, e.g., Lebesgue's dominated convergence theorem). Moreover, for any $\phi \in C_c^1(a,b)$, we have

$$\begin{aligned}
\int_a^b \tilde{u}(x)\varphi'(x)dx &= \int_a^y \int_y^x u'(t)\varphi'(x)dtdx + \int_y^b \int_y^x u'(t)\varphi'(x)dtdx \\
\text{(by Fubini)} &= -\int_a^y u'(t)\left(\int_a^t \varphi'(x)dx\right)dt + \int_y^b u'(t)\left(\int_t^b \varphi'(x)dx\right)dt \\
&= -\int_a^b u'(t)\varphi(t)dt
\end{aligned}$$

Hence $\tilde{u}' \in L^p(a,b)$ and $\tilde{u}' = u'$ a.e. in (a,b), i.e. $u = \tilde{u} + c$ a.e. Since $\tilde{u}$ is continuous, we conclude that u admits a continuous representative on $[a,b]$. Moreover, for any $x_1, x_2 \in (a,b)$,

$$\begin{aligned}
|u(x_1) - u(x_2)| = |\tilde{u}(x_1) - \tilde{u}(x_2)| &= \left|\int_{x_2}^{x_1} u'(t)dt\right| \\
&\leq \left(\int_{x_2}^{x_1} |u'(t)|^p dt\right)^{1/p} |x_1 - x_2|^{1-1/p} \leq \|u'\|_{L^p} |x_1 - x_2|^{1-1/p}
\end{aligned}$$

and

$$\begin{aligned}
|u(x)| &= \left|\frac{1}{b-a}\int_a^b u(x)dy\right| \leq \frac{1}{b-a}\int_a^b |u(y)|dy + \frac{1}{b-a}\int_a^b |u(x)-u(y)|dy \\
&\leq (b-a)^{-\frac{1}{p}}\|u\|_{L^p} + (b-a)^{1-\frac{1}{p}}\|u'\|_{L^p} \leq C\|u\|_{W^{1,p}}
\end{aligned}$$

Hence, there exists $C > 0$ such that

$$\begin{aligned}
&\|u\|_{C^0([a,b])} \leq C\|u\|_{W^{1,p}} \\
&\|u\|_{C^{0,1-1/p}([a,b])} = \|u\|_{C^0([a,b])} + \sup_{x_1,x_2\in(a,b)} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^{1-1/p}} \leq C\|u\|_{W^{1,p}}
\end{aligned}$$

and $u \in C^{0,1-1/p}([a,b])$ and, more generally, $u \in C^{0,\alpha}([a,b])$ for any $\alpha \in [0, 1-1/p]$.

Exercise 2. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is an integrable function with compact support such that $\int_{\mathbb{R}} g(t)\,\mathrm{d}t = 0$. Define

$$f(x) = \int_{-\infty}^x g(t)\,\mathrm{d}t,$$

and prove that

$$|f(x)| \leq \frac{1}{2}\int_{\mathbb{R}} |g(t)|\,\mathrm{d}t.$$

Hint: Let $g = g_+ - g_-$, where the non-negative functions g_+, g_- are defined by $g_+ := \max(g, 0)$, $g_- := \max(-g, 0)$.

Solution: See Lecture Notes.

Exercise 3. Suppose that $n \geq 2$ and $g_i \in C_c^\infty\left(\mathbb{R}^{n-1}\right)$, for $1 \leq i \leq n$, are non-negative functions. Define $g \in C_c^\infty\left(\mathbb{R}^n\right)$ by

$$g(x) := \prod_{i=1}^{n} g_i\left(x_i'\right)$$

Prove that

$$\int_{\mathbb{R}^n} g \,\mathrm{d}x \leq \prod_{i=1}^{n} \|g_i\|_{L^{n-1}(\mathbb{R}^n)} .$$

Hint: See the lecture notes.

Solution: See Lecture Notes.