Exercise Sheet 8

Introduction to Partial Differential Equations (W. S. 2024/25)

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- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

Exercise 1. Let $\Omega = B_1(0) \subset \mathbb{R}^n$, $n \geq 1$.

(i) For which combinations $(\alpha, p) \in (\mathbb{R}\backslash\{0\}) \times [1, \infty]$ does the function $|x|^\alpha$ belong to $L^p(\Omega)$ and/or to $W^{1,p}(\Omega)$?

(ii) For which values $1 \leq p \leq \infty$ does $\log|x|$ belong to $L^p(\Omega)$ and/or to $W^{1,P}(\Omega)$?

(iii) Comment on the local (Ω is bounded) L^p and $W^{1,p}$ regularity of the fundamental solution of the Laplacian Φ in $\mathbb{R}^n, n \geq 2$.

Solution:

(i) Since $u_\alpha := |x|^\alpha \in C^\infty(\bar{\Omega}\backslash\{0\})$, the weak derivative, if it exists, must coincide with the classical one:

$$Du_\alpha(x) = \alpha \frac{x}{|x|}|x|^{\alpha-1} = \alpha \frac{x}{|x|} u_{\alpha-1}(x)$$

Let us consider $p = \infty$ first. We have $u_\alpha \in L^\infty(\Omega)$ iff $\alpha > 0$, and $u_\alpha \in W^{1,\infty}(\Omega)$ iff $\alpha \geq 1$.

Now, let $1 \leq p < \infty$. We compute explicitly

$$\|u_\alpha\|^p_{L^p(\Omega)} = \int_0^1 \left(\int_{\partial B_r} 1\right) r^{p\alpha}\, \mathrm{d}r = \omega_n \int_0^1 r^{p\alpha+n-1}\, \mathrm{d}r$$

(for $n = 1$ the computation becomes formal, but is still correct, with $\omega_1 := 2$), which is bounded iff $p\alpha + n - 1 > -1$. This implies that $u_\alpha \in L^p(\Omega)$ iff $p\alpha + n > 0$. Replacing α with $\alpha - 1$, we conclude that $u_\alpha \in W^{1,p}(\Omega)$ iff $p(\alpha - 1) + n > 0$.

(ii) We proceed similarly for v, whose weak derivative, if it exists, is equal to

$$Dv(x) = \frac{x}{|x|^2} = \frac{x}{|x|} u_{-1}(x),$$

with u_α defined as in (i).

First, we note that v is not bounded, so it does not belong to $L^\infty(\Omega)$. We can compute

$$\|v\|^p_{L^p(\Omega)} = \int_0^1 \left(\int_{\partial B_r} 1 \right) (-\log r)^p \, \mathrm{d}r = \omega_n \int_0^1 r^{n-1}(-\log r)^p \, \mathrm{d}r < \infty$$

for all $n \geq 1$ and $1 \leq p < \infty$. From the results on u_{-1}, we know that $Dv \in L^p(\Omega)$ iff $-p+n > 0$. In conclusion, $v \in L^p(\Omega)$ iff $1 \leq p < \infty$ and $v \in W^{1,p}(\Omega)$ iff $1 \leq p < n$.

(iii) Now, consider the fundamental solution of the Laplacian in $\mathbb{R}^n$:

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & \text{if } n = 2, \\ \frac{1}{(n-2)\omega_n} |x|^{2n} & \text{if } n \geq 3. \end{cases}$$

For $n = 2, \Phi \in L^p(\Omega)$ iff $1 \leq p < \infty$ and $\Phi \in W^{1,p}(\Omega)$ iff $1 \leq p < 2$.

For $n \geq 3, \Phi \in L^p(\Omega)$ iff $1 \leq p < \frac{n}{n-2}$ and $\Phi \in W^{1,p}(\Omega)$ iff $1 \leq p < \frac{n}{n-1}$.

In particular, we remark that $\Phi \in L^2(\Omega)$, yet $\Phi \notin H^1(\Omega)$ for $n = 2, 3$, and $\Phi \notin L^2(\Omega)$ for $n \geq 4$.

Exercise 2. Let $\Omega \subset \mathbb{R}^n$ be a domain, and $u \in W^{1,p}(\Omega)$ for some $1 \leq p < \infty$. Consider an arbitrary $f \in C^1(\mathbb{R})$, with f' bounded.

(i) Let Ω be bounded. Show that $f \circ u := f(u(\cdot)) \in W^{1,p}(\Omega)$ and that $D(f \circ u) = f'(u)Du$.

(ii) Assuming $f(0) = 0$, extend the chain rule above to the case where Ω is unbounded.

(iii) Show that $u^+ = \max\{u, 0\}, u^- = \min\{u, 0\}$, and $|u|$ belong to $W^{1,p}(\Omega)$ as well.

Hint: use appropriately mollified versions of the functions $\max\{\cdot, 0\}$, etc.

Solution:

(i) We first show that $f \circ u \in L^p(\Omega)$ since,

$$\begin{aligned} \|f(u)\|^p_{L^p(\Omega)} &= \int_\Omega \left| f(0) + u \int_0^1 f'(0 + s(u-0)) \, \mathrm{d}s \right|^p \\ &\leq 2^{p-1} \int_\Omega |f(0)|^p + 2^{p-1} \int_\Omega |u|^p \left(\int_0^1 |f'(su)| \, \mathrm{d}s \right)^p \\ &\leq 2^{p-1} |\Omega| |f(0)|^p + 2^{p-1} \|u\|^p_{L^p(\Omega)} \|f'\|^p_{L^\infty(\mathbb{R})} < \infty. \end{aligned}$$

We now aim to show that the weak derivative of $f \circ u$ is in $L^p(\Omega)$ and that $D(f \circ u) = f'(u)Du$. To do so, we recall that there exists a regularized family of function u_m such that u_m converges to u in $L^p(\Omega)$, and for any compact set $K \subset\subset \Omega, u_m \to u$ in $W^{1,p}(K)$, that is,

$$\|u_m - u\|_{L^p(K)} + \|Du_m - Du\|_{L^p(K)} \to 0.$$

Up to subsequences, we can also assume that $u_m \to u$ a.e. in Ω. Then we note that $f \circ u_m \to f \circ u$ in $L^p(\Omega)$:

$$
\begin{aligned}
\|f(u_m) - f(u)\|_{L^p(\Omega)} &= \left(\int_\Omega \left| (u_m - u) \int_0^1 f'(u + s(u_m - u))\, \mathrm{d}s \right|^p \right)^{1/p} \\
&\leq \|f'\|_{L^\infty(\mathbb{R})} \|u_m - u\|_{L^p(\Omega)} \to 0
\end{aligned}
$$

We next show that the weak (classical) derivative of $f \circ u_m$ tends to $f'(u)Du$ in $L^p(K)$ for every compact subset K. Indeed,

$$
\begin{aligned}
\|f'(u_m) Du_m - f'(u)Du\|_{L^p(K)} &\leq \|f'(u_m)(Du_m - Du)\|_{L^p(K)} + \|(f'(u_m) - f'(u)) Du\|_{L^p(K)} \\
&\leq \|f'\|_{L^\infty(K)} \underbrace{\|Du_m - Du\|_{L^p(K)}}_{\longrightarrow 0} + \left(\int_K |f'(u_m) - f'(u)|^p |Du|^p \right)^{1/p}.
\end{aligned}
$$

Since $u_m \to u$ a.e. in Ω and f' is continuous, we have $f'(u_m) \to f'(u)$ a.e. in Ω. We then invoke the dominated convergence theorem and see that the last integral converges to 0. Hence, $Df(u_m) \to f'(u)Du$ in $L^p(K)$. To conclude the proof, we show that $D(f \circ u) = f'(u)Du$. Now, observe that, for every $\phi \in C_0^\infty(\Omega)$ such that supp $(\phi) \subset K$, $L^p(K)$-convergence of v_m to v implies convergence (in $\mathbb{R}$) of $\int_K v_m \phi$ to $\int_K v\phi$:

$$
\left| \int_K v_m \phi - \int_K v\phi \right| \leq \int_K |v_m - v|\, |\phi| \leq \|v_m - v\|_{L^p(K)} \|\phi\|_{L^q(K)} \to 0,
$$

where $p^{-1} + q^{-1} = 1$ (by Hölder's inequality). Therefore for every ϕ compactly supported in K and, due to the arguments above,

$$
\begin{aligned}
-\int_\Omega f(u) D\phi &= -\lim_{m\to\infty} \int_K f(u_m) D\phi = \lim_{m\to\infty} \int_K D(f(u_m))\phi = \lim_{m\to\infty} \int_K f'(u_m) Du_m \phi \\
&= \int_\Omega f'(u) Du \phi
\end{aligned}
$$

Since f' is bounded and $f'(u)Du \in L^p(\Omega)$, the proof is completed.

(ii) We can apply the same reasoning as above. The only difference is that, thanks to the hypothesis $f(0) = 0$, $f \circ u_m$ and $f \circ u$ belongs to $W^{1,p}(\Omega)$:

$$
\|f(u_m)\|_{L^p(\Omega)}^p = \int_\Omega \left| f(0) + u_m \int_0^1 f'(su_m)\, \mathrm{d}s \right|^p \leq \|u_m\|_{L^p(\Omega)}^p \|f'\|_{L^\infty(\mathbb{R})}^p,
$$

and

$$
\|Df(u_m)\|_{L^p(\Omega)}^p = \int_\Omega |f'(u_m) Du_m|^p \leq \|f'\|_{L^\infty(\mathbb{R})}^p \|Du_m\|_{L^p(\Omega)}^p < \infty.
$$

(iii) Let

$$
f_\varepsilon(t) = \left(\sqrt{t^2 + \varepsilon^2} - \varepsilon \right) \mathbb{1}_{\mathbb{R}_+}(t), \quad f_\varepsilon'(t) = t\left(t^2 + \varepsilon^2\right)^{-1/2} \mathbb{1}_{\mathbb{R}_+}(t).
$$

Then, we have $f_\varepsilon(0) = 0, f_\varepsilon \in C^1(\mathbb{R}), f_\varepsilon(t) \to \max\{0, t\}$ as $\varepsilon \to 0$ for each $t \in \mathbb{R}$, and

$|f_\varepsilon'| < 1$ independently of ε. We apply (i) and (ii) to obtain $Df_\varepsilon(u) = f_\varepsilon'(u)Du$, i.e., for all $\phi \in C_c^\infty(\Omega)$,

$$-\int_\Omega f_\varepsilon(u) D\phi = \int_{\{u(x)>0\}} \frac{u(x)}{\left(u(x)^2+\varepsilon^2\right)^{1/2}} Du(x)\phi(x)\,\mathrm{d}x$$

Now, noting that we have $|f_\varepsilon(u)| \le |u|$, it suffices to apply the dominated/monotone convergence theorem (for $\varepsilon \to 0$) on both sides to conclude that $Du^+ = \mathbb{1}_{\{u>0\}}Du \in L^p(\Omega)$ a.e. in Ω. The second claim follows from $u^- = -(-u)^+$, and the third from $|u| = u^+ - u^-$.

Exercise 3. The application $\|\cdot\|_{k,p} : W^{k,p}(\Omega) \to \mathbb{R}_+$ (defined in the Lecture Notes) is a norm for any $1 \le p \le \infty$.

Solution: See Lecture Notes.

Exercise 4. The normed vector space $\left(W^{k,p}(\Omega), \|\cdot\|_{k,p}\right)$ is a Banach space for every $k \in \mathbb{N}$ and $1 \le p \le \infty$. In particular, the space $H^k(\Omega) = W^{k,2}(\Omega)$ is a Hilbert space, for every $k \in \mathbb{N}$, with inner product

$$(f,g)_{H^k} := \sum_{|\alpha|\le k} \int_\Omega D^\alpha f \cdot D^\alpha g\,\mathrm{d}x.$$

Solution: See Lecture Notes.

Exercise 5. Let $f \in W^{k,p}(\Omega)$, with $1 \le p < \infty$, and $f_\epsilon := \eta_\epsilon * f : \Omega \to \mathbb{R}$. Then $f_\epsilon \xrightarrow{\epsilon\to 0} f$ in $L^p(\Omega)$ and $f_\epsilon \xrightarrow{\epsilon\to 0} f$ in $W^{k,p}(K)$, for any $K \subset\subset \Omega$.

Solution: See Lecture Notes.