Exercise Sheet 7
Introduction to Partial Differential Equations (W. S. 2024/25)
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- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

Exercise 1. Let us start by recalling the classic Ascoli–Arzelà theorem.[1]

> Let X be a compact metric space. If a sequence $\{f_n\}_{n=1}^{\infty}$ in the space[2] $C(X)$ is bounded[3] and equi-continuous[4], then it has a uniformly convergent subsequence. Moreover, if every subsequence of $\{f_n\}_{n=1}^{\infty}$ itself has a uniformly convergent subsequence, then $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded and equi-continuous.

Keeping Ascoli–Arzelà's theorem in mind, prove the following result.

> *Ascoli–Arzelà-type theorem for harmonic functions.* If $\{u_m\}_{m=1}^{\infty}$ is a sequence of harmonic functions on Ω that is uniformly bounded on each compact subset of Ω, then some subsequence of $\{u_m\}_{m=1}^{\infty}$ converges uniformly on each compact subset of Ω.

Solution: The key to the proof is the following observation: there exists a constant $C < \infty$ such that for all u harmonic and bounded by M on any ball $B_{2r}(a)$,

$$|u(x) - u(a)| \leq \left(\sup_{B_r(a)} |\nabla u| \right) |x - a| \leq \frac{CM}{r} |x - a|$$

[1] Proven first by Giulio Ascoli [Asc84] (who established the sufficient condition for compactness) and then by Cesare Arzelà [Arz95] (who established also the necessary condition). The generalization to real-valued continuous functions with domain a compact metric space is due to Maurice Fréchet [Fré06].

[2] Our setting is a compact metric space X which you can, if you wish, take to be a compact subset of $\mathbb{R}^n$. Let $C(X)$ denote the space of all continuous functions on X with values in $\mathbb{R}$. In $C(X)$ we always regard the distance between functions f and g in $C(X)$ to be

$$\operatorname{dist}(f, g) := \max\{|f(x) - g(x)| : x \in X\}.$$

It is easy to check that dist, so defined, is a metric (the *max-metric*) on $C(X)$, in which a sequence is convergent iff it converges uniformly on X. Similarly, a sequence in $C(X)$ is Cauchy iff it is Cauchy uniformly on X. Thus the max-metric, which from now on we always assume to be part of the definition of $C(X)$, makes that space complete

[3] A family $\mathcal{F} \subset C(X)$ being *bounded* means that there exists a positive constant $M < \infty$ such that $|f(x)| \leq M$ for each $x \in X$ and each $f \in \mathcal{F}$

[4] The family $\mathcal{F} \subset C(X)$ being *equi-continuous* means that for every $\varepsilon > 0$ there exists $\delta > 0$ (which depends only on ε) such that, for $x, y \in X$,

$$d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \qquad \text{for all } f \in \mathcal{F}$$

where d is the metric on X.

for all $x \in B_r(a)$. The first inequality is standard from advanced calculus; the second inequality follows from the estimate on the derivatives of harmonic functions proved in the Lecture Notes.

Now suppose $K \subset \Omega$ is compact, and let $r = d(K, \partial\Omega)/3$. Because the set $K_{2r} = \{x \in \mathbb{R}^n : d(x, K) \le 2r\}$ is a compact subset of Ω, the sequence $\{u_m\}$ is uniformly bounded by some $M < \infty$ on K_{2r}. Let $a, x \in K$ and assume $|x - a| < r$. Then $x \in B_r(a)$ and $|u_m| \le M$ on $B_{2r}(a) \subset K_{2r}$ for all m, and so we conclude from the first paragraph that

$$|u_m(x) - u_m(a)| \le \frac{CM}{r}|x - a|$$

for all m. It follows that the sequence $\{u_m\}$ is equicontinuous on K. To finish, choose compact sets

$$K_1 \subset K_2 \subset \cdots \subset \Omega$$

whose interiors cover Ω. Because $\{u_m\}$ is equicontinuous on K_1, the Ascoli–Arzelà theorem recalled above implies $\{u_m\}$ contains a subsequence that converges uniformly on K_1. Applying Ascoli–Arzelà again, there is a subsequence of this subsequence converging uniformly on K_2, and so on. If we list these subsequences one after another in rows, then the subsequence obtained by traveling down the diagonal converges uniformly on each K_j, and hence on each compact subset of Ω.

Exercise 2. Consider the Neumann problem for the Laplace equation on the half space $\Omega := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$:

$$\begin{cases} -\Delta u = 0, & x \in \Omega, \\ \partial_\nu u = h, & x \in \partial\Omega, \end{cases}$$

with $h \in C_0^2(\partial\Omega)$. Let $N(x, y) := \Phi(x - y) + \Phi\left(x - y^\star\right)$, where $y^\star := (y_1, \ldots, y_{n-1}, -y_n)$.

(i) Prove that $u(y) = \int_{\partial\Omega} N(x, y)h(x)\,\mathrm{d}x$ is well-defined for all $y \in \Omega$, and satisfies $-\Delta u = 0$ in Ω and $\partial_\nu u = h$ on $\partial\Omega$.

(ii) Prove that for $n \ge 3$ we have $\lim_{|y|\to\infty} u(y) = 0$.

Solution: For $x \in \partial\Omega$ and $y \in \Omega$ we have

$$\begin{aligned} N(x, y) &= \Phi\left(\sqrt{\sum_{j=1}^{n-1}(x_j - y_j)^2 + (-y_n)^2}\right) + \Phi\left(\sqrt{\sum_{j=1}^{n-1}(x_j - y_j)^2 + (y_n)^2}\right) \\ &= 2\Phi(x - y) \end{aligned}$$

and thus

$$\begin{aligned} |u(y)| = \left|\int_{\partial\Omega} 2\Phi(\cdot - y)h\right| &\le 2\sup_{\partial\Omega}|h| \int_{B_R(0)} \left|\Phi\left(\sqrt{\sum_{j=1}^{n-1}(\tilde{x}_j - y_j)^2 + y_n^2}\right)\right| \mathrm{d}\tilde{x} \\ &= \begin{cases} c_2 \sup_{\partial\Omega}|h|\,|B_R(0)| \max_{|x_1|\le R}\left|\ln\left(\sqrt{(x_1 - y_1)^2 + y_2^2}\right)\right| & n = 2 \\ c_n \sup_{\partial\Omega}|h|\,|B_R(0)|\, r_y^{2-n} & n \ge 3 \end{cases} \end{aligned}$$

where R is large enough so that $\operatorname{supp}(h) \subset B_R(0) \subset \mathbb{R}^{n-1}$, and

$$r_y := \min_{\tilde{x} \in \in} \sqrt{\sum_{j=1}^{n-1} (\tilde{x}_j - y_j)^2 + y_n^2} > 0.$$

Thus, $u(y)$ is well defined. For $n \geq 3$, noting that as $|y| \to \infty, r_y \to \infty$, so we have $u(y) \to 0$ as $|y| \to \infty$, which establishes the statement (ii).

For each fixed $x \in \mathbb{R}^n$, the mapping $y \mapsto N(x,y)$ is harmonic, except for $x = y$. In particular, for $x \in \partial\Omega, N(x,\cdot)$ is harmonic in Ω. Together with the compactness of $\operatorname{supp}(h)$, using the dominated convergence theorem we have $\Delta u(y) = \int_{\partial\Omega} h \Delta_y N(\cdot, y) = 0$ for any $y \in \Omega$. Moreover, for $y \in \Omega$ we have

$$\partial_\nu u(y) = -\partial_{y_n} u(y) = -\partial_{y_n} \int_{\partial\Omega} N(\cdot, y) h = -2 \int_{\partial\Omega} h \partial_{y_n} \Phi(\cdot - y) = \frac{2}{\omega_n} \int_{\partial\Omega} \frac{h y_n}{|\cdot - y|^n}$$

which is nothing but the Poisson integral formula in the half plane. Arguing as in Problem 4 of Exercise Sheet 6, we conclude

$$\lim_{\substack{y \to y^0 \\ y \in \mathbb{R}^n_+}} -\partial_{y_n} u(y) = h\left(y^0\right) \qquad \text{for each } y^0 \in \partial\mathbb{R}^n_+.$$

Exercise 3. Let $n \geq 2$ and $f \in C_0^2(\mathbb{R}^n)$. Given Φ the fundamental solution of the Laplace equation in $\mathbb{R}^n$, consider

$$w(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) \, \mathrm{d}y.$$

(i) Use the change of variable $y = x + z$ to rewrite this quantity as $w(x) = \int_{\mathbb{R}^n} \Phi(z) f(x+z) \, \mathrm{d}z$. Show that $w \in C^2(\mathbb{R}^n)$ and $\partial^2_{x_i x_j} w(x) = \int_{\mathbb{R}^n} \Phi(z) \partial_{x_i x_j} f(x+z) \, \mathrm{d}z$.

(ii) Prove that $-\Delta w = f$ in $\mathbb{R}^n$.

(iii) Show that $\lim_{|x| \to \infty} w(x) = 0$ if $n \geq 3$.

Solution:

(i) We need a few ingredients:

(a) (dominated convergence theorem) given a sequence of measurable functions $\phi_m : \Omega \to \mathbb{R}$ converging pointwise to $\phi : \Omega \to \mathbb{R}$ as $m \to \infty$, if there exists a measurable function $\psi : \Omega \to \mathbb{R}$ such that $|\phi_m| \leq \psi$ in Ω and $\int_\Omega |\psi| < \infty$, then $\int_\Omega |\phi| < \infty$, and $\int_\Omega |\phi_m - \phi| \to 0$;

(b) (dominated convergence theorem for derivatives) given a measurable function $\phi : \mathbb{R}^n \times \mathbb{R}^n \supset \Omega' \times \Omega \to \mathbb{R}, \phi : (x,y) \mapsto \phi(x,y)$, such that - $\phi(x,\cdot) : \Omega \to \mathbb{R}$ is integrable for every $x \in \Omega'$; - $\partial_{x_i} \phi(x,y)$ exists for every $(x,y) \in \Omega' \times \Omega$; - there exists a Lebesgue integrable function $\psi : \Omega \to \mathbb{R}$ such that $|\partial_{x_i}\phi(x,\cdot)| \leq \psi$ in Ω for every $x \in \Omega'$; then $\partial_{x_i}\int_\Omega \phi(x,\cdot) = \int_\Omega \partial_{x_i}\phi(x,\cdot)$ for all $x \in \Omega'$. To prove this, it suffices to fix $x \in \Omega'$ arbitrarily and apply the dominated convergence theorem to the quotient ratio $q_m(y) = \frac{\phi(x+h_m e_i,y)-\phi(x,y)}{h_m}$ for $\mathbb{R}\backslash\{0\} \in h_m \to 0$. Note that q_m can be bounded by ψ independently of m :

$$|q_m| = \left|\partial_{x_i}\phi\left(x+\theta_{x,m,y}e_i,\cdot\right)\right| \leq \psi$$

where $\theta_{x,m,y}$ is the constant from the mean value theorem.

(c) (local integrability of Φ) it holds that

$$|\Phi(r)| = \begin{cases} -\frac{1}{2\pi}|\log r| & \text{for } n=2 \\ \frac{r^{2-n}}{(n-2)\omega_n} & \text{for } n \geq 3 \end{cases} \leq c_n(|\log r|+1)r^{2-n}$$

so that, given $R>0$,

$$\begin{aligned}\int_{B_R}|\Phi| &= \int_0^R\left(\int_{\partial B_r}1\right)|\Phi(r)|\mathrm{d}r \leq c_n\omega_n\int_0^R r^{n-1}(|\log r|+1)r^{2-n}\,\mathrm{d}r \\ &= c_n\omega_n\int_0^R r(|\log r|+1)\mathrm{d}r =: C_{n,R} < \infty\end{aligned}$$

Now we are ready to start. The first step is rewriting w as

$$w(x) := \int_{\mathbb{R}^n}\Phi(y)f(x+y)\mathrm{d}y = \int_{\Omega_x}\Phi(y)f(x+y)\mathrm{d}y$$

where $\Omega_x = \{y - x \mid y \in \operatorname{supp} f\}$ is bounded for all x. (We do this to exploit the smoothness of f. The argument below cannot be applied if x is in the argument of Φ, e.g. because $\Delta\Phi$ is not locally integrable around 0 for any $n \geq 2$.) Let x be fixed, and define

$$\phi_m(y) = \Phi(y)f(x+y)1_{\Omega_x\backslash B_{\frac{1}{m}}}(y)$$

The sequence ϕ_m converges pointwise to $\Phi(\cdot)f(x+\cdot)$, and its absolute value is bounded from above by the non-negative function

$$\psi(y) = c_n(|\log|y||+1)|y|^{2-n}\max_{\mathbb{R}^n}|f|1_{\Omega_x}(y)$$

independently of m. In particular, given $R_x := \max_{y\in\Omega_x}|y|$, owing to (c) it holds that

$$\int_{\mathbb{R}^n}\psi \leq \int_{B_{R_x}}\psi \leq C_{n,R_x}\max_{\mathbb{R}^n}|f| < \infty$$

Thus, (a) implies that w is well-defined for all x. Now we wish to apply (b) twice to show the existence of all second order partial derivatives of w. To do this, we need to bound from above $|\Phi(y)\partial_{x_i}f(\tilde{x}+y)|$ and $\left|\Phi(y)\partial_{x_ix_j}f(\tilde{x}+y)\right|$, uniformly in $\tilde{x}$ on Ω_x. Here we show the strategy only for $|\Phi(y)\partial_{x_1}f(x+y)|$: thanks to (iii),

$$|\Phi(y)\partial_{x_1}f(x+y)| \leq \psi(y) = c_n(|\log|y||+1)|y|^{2-n}\max_{\mathbb{R}^n}|\partial_{x_1}f|\,\mathbb{1}_{\Omega_x}(y)$$

and (b) can be applied. It just remains to show that all second derivatives $\partial_{x_i x_j} w(x) = \int_{\mathbb{R}^n} \Phi(y) \partial_{x_i x_j} f(x+y) \mathrm{d}y$ are continuous in x. We can check this by using the uniform continuity of $\partial_{x_i x_j} f$ on compact sets and the local integrability of Φ.

(ii) Let $R_0 = \max_{y \in \operatorname{supp} f} |y|$, so that $\operatorname{supp} f \subset B_{R_0}$. We have shown in the previous point that $\Delta w(x) = \int_{\mathbb{R}^n} \Phi(x-y) \Delta f(y) \mathrm{d}y = \int_{B_{R_0}} \Phi(x-y) \Delta f(y) \mathrm{d}y$. But then, the Green's representation formula together with $\operatorname{supp} f \subset B_{R_0}$ implies

$$
\begin{aligned}
-\int_{B_{R_0}} \Phi(x-y) \Delta f(y) \mathrm{d}y &= f(x) + \int_{\partial B_{R_0}} \left(\partial_\nu \Phi(x-y) f(y) - \Phi(x-y) \partial_\nu f(y) \right) \mathrm{d}S(y) \\
&= f(x) + 0
\end{aligned}
$$

and thus $-\Delta w = f$.

(iii) We use the compactness of the support of f and the monotonicity of $|\Phi|$: for all $x \in \mathbb{R}^n$ such that $|x| > R_0$, we have $|x-y| > 0$ for any $y \in \operatorname{supp} f$ (i.e. $\Phi(x-y)$ is well-behaved in B_{R_0}), and

$$
\begin{aligned}
|w(x)| &\leq \int_{B_{R_0}} |\Phi(x-y) f(y)| \mathrm{d}y \leq |B_{R_0}| \frac{\max}{B_{R_0}} |\Phi(x-\cdot)| \max_{\mathbb{R}^n} |f| \\
&= |B_{R_0}| \, |\Phi(r_x)| \max_{\mathbb{R}^n} |f| = c_n \, |B_{R_0}| \max_{\mathbb{R}^n} |f| r_x^{2-n}
\end{aligned}
$$

with $r_x = \min_{y \in \overline{B_{R_0}}} |x-y| > 0$. The claim follows, since, by the triangular inequality, $r_x > |x| - R_0 \to \infty$ as $|x| \to \infty$.

Exercise 4. Let Ω be a bounded domain and $f \in C_c^2(\Omega)$. Then, the Newtonian potential

$$
w(y) = \int_\Omega \Phi(x-y) f(x) \, \mathrm{d}x, \qquad y \in \mathbb{R}^n,
$$

of f satisfies $w \in C^2(\mathbb{R}^n)$ and $-\Delta w = f$ in Ω.

Solution: Since f has compact support in Ω, its extension by zero outside Ω, which we denote $\tilde{f}$, is still a C^2 function. We can then rewrite the Newtonian potential as

$$
w(y) = \int_\Omega \Phi(x-y) f(x) \, \mathrm{d}x = \int_{\mathbb{R}^n} \Phi(x-y) \tilde{f}(x) \, \mathrm{d}x = \int_{\mathbb{R}^n} \Phi(z) \tilde{f}(z+y) \, \mathrm{d}z.
$$

Since $\tilde{f} \in C_c^2(\mathbb{R}^n)$ and Φ is locally integrable we have that $w \in C^2(\mathbb{R}^n)$ and

$$
-\Delta w(y) = -\int_{\mathbb{R}^n} \Phi(z) \Delta \tilde{f}(z+y) \, \mathrm{d}z = -\int_{B_R(0)} \Phi(x-y) \Delta \tilde{f}(x) \, \mathrm{d}x
$$

where $B_R(0)$ is any sufficiently large ball that contains Ω. Using now Green's representation formula we have

$$
\tilde{f}(y) = \int_{B_R(0)} \Phi(x-y)(-\Delta \tilde{f}(x)) \, \mathrm{d}x - \int_{\partial B_R(0)} \partial_\nu \Phi(x-y) \tilde{f}(x) \, \mathrm{d}S(x) + \int_{\partial B_R(0)} \Phi(x-y) \partial_\nu \tilde{f}(x) \, \mathrm{d}S(x)
$$

and the fact that $\tilde{f} \in C_c^2(B_R(0))$, we conclude that $-\Delta w(y) = \tilde{f}(y) = f(y)$ for all $y \in \Omega$.

References

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