Exercise Sheet 6

Introduction to Partial Differential Equations (W. S. 2024/25)

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- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

Exercise 1. Let $T \in \mathcal{D}'(\mathbb{R})$. Prove that $T' = 0$ if and only if T is constant.

Hint: If we assume that T is constant, then we note that $T' = 0$ because φ has compact support. Conversely, suppose that $T' = 0$. Then, for all $\varphi \in C_c^\infty(\mathbb{R})$, $\langle T', \varphi \rangle = -\langle T, \varphi' \rangle = 0$, i.e., T vanishes on all functions of the form φ', where $\varphi \in C_c^\infty(\mathbb{R})$. To prove the result, it is helpful to characterize such functions. In particular, one should prove that

$$\big(\psi = \varphi', \text{ with } \varphi \in C_c^\infty(\mathbb{R})\big) \iff \left(\psi \in C_c^\infty(\mathbb{R}) \text{ and } \int_{\mathbb{R}} \psi(x)\,\mathrm{d}x = 0\right).$$

Solution: See Proposition 3.4 of the Lecture Notes.

Exercise 2. Let $\Gamma : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be defined by $\Gamma(x) = \frac{1}{8\pi}|x|^2 \log|x|$.

(i) Show that $\Delta^2\Gamma \coloneqq \Delta(\Delta\Gamma)$ is equal to 0 in $\mathbb{R}^2 \setminus \{0\}$. Moreover, observe that $\int_{\partial B_r(0)} \partial\nu \Delta\Gamma(x)\,\mathrm{d}S(x) = 1$ for all $r > 0$.

(Extra point) Show that Γ is the fundamental solution of the bilaplacian operator Δ^2.

(ii) Let $u \in C^4(\bar{\Omega})$, with $\Omega \subset \mathbb{R}^2$ an open bounded domain with smooth boundary. Prove that, for all $x \in \Omega$, the following Green-type representation formula holds:

$$u(x) = \int_\Omega \Gamma(x - \cdot)\Delta^2 u - \int_{\partial\Omega} \left(\Gamma(x - \cdot)\partial_\nu \Delta u - \Delta u \partial_\nu \Gamma(x - \cdot) + \Delta\Gamma(x - \cdot)\partial_\nu u - u\partial_\nu \Delta\Gamma(x - \cdot)\right).$$

Solution:

(i) Let $r = |x|$, so that $\Gamma(x) = \Gamma(r)$. Then $\Gamma'(r) = (2r\log r + r)/8\pi$ and $\Gamma''(r) = (2\log r + 3)/8\pi$. Then

$$\Delta\Gamma(x) = \Gamma''(r) + \frac{1}{r}\Gamma'(r) = \frac{1}{8\pi}(2\log r + 3 + 2\log r + 1) = \frac{1}{2\pi}(\log r + 1)$$

and

$$\Delta^2\Gamma(x) = (\Delta\Gamma)''(r) + \frac{1}{r}(\Delta\Gamma)'(r) = \frac{1}{2\pi}\left(-\frac{1}{r^2} + \frac{1}{r^2}\right) = 0$$

It remains to verify that the normalization constant is such that

$$\int_{\partial B_r(0)} \partial_\nu \Delta\Gamma(x)\,\mathrm{d}S(x) = 1 \qquad \text{for all } r > 0$$

But $\partial_\nu \Delta\Gamma(x) = (\Delta\Gamma)'(x) = 1/(2\pi|x|) = 1/\int_{\partial B_r(0)} 1$ for $x \in \partial B_r(0)$, and the result follows.

(ii) Given $\varepsilon > 0$, we employ the second Green's identity in $\Omega_c = \Omega \setminus B_\varepsilon(x)$

$$\int_{\Omega_\varepsilon} (\Delta f g - f\Delta g) = \int_{\delta\Omega_\varepsilon} (g\partial_\nu f - f\partial_\nu g)$$

with $(f,g) = (\Delta\Gamma(x-\cdot), u)$ and $(f,g) = (\Gamma(x-\cdot), \Delta u)$, respectively (both u and $\Gamma(x-\cdot)$ are in $C^4(\Omega_\kappa))$:

$$\begin{cases} \int_{\Omega_\varepsilon} (\underbrace{\Delta^2\Gamma(x-\cdot)}_{0} u - \Delta\Gamma(x-\cdot)\Delta u) = \int_{\partial\Omega_\varepsilon} (u\partial_\nu \Delta\Gamma(x-\cdot) - \Delta\Gamma(x-\cdot)\partial_\nu u) \\ \int_{\Omega_\varepsilon} \left(\Delta\Gamma(x-\cdot)\Delta u - \Gamma(x-\cdot)\Delta^2 u\right) = \int_{\partial\Omega_\varepsilon} (\Delta u \partial_\nu \Gamma(x-\cdot) - \Gamma(x-\cdot)\partial_v \Delta u) \end{cases}$$

Adding up the two equations yields

$$\begin{aligned} \underbrace{\int_{\Omega_\varepsilon} \Gamma(x-\cdot)\Delta^2 u}_{=:I_0} = \underbrace{\int_{\partial\Omega_\varepsilon} \Gamma(x-\cdot)\partial_\nu\Delta u}_{=:I_1} - \underbrace{\int_{\partial\Omega_\varepsilon} \Delta u\partial_\nu\Gamma(x-\cdot)}_{=:I_2} \\ + \underbrace{\int_{\Omega_\varepsilon} \Delta\Gamma(x-\cdot)\partial_\nu u}_{=:I_3} - \underbrace{\int_{\Omega_\varepsilon} u\partial_\nu\Delta\Gamma(x-\cdot)}_{=:I_4} \end{aligned}$$

Since we have $u \in C^4(\bar{\Omega})$, and $\Gamma(x-\cdot)$, $\partial_\nu\Gamma(x-\cdot) \in L^1_{\mathrm{loc}}(\Omega)$, we have that I_0, I_1, and I_2 converge to the respective integrals over Ω as $\varepsilon \to 0$. We only need to deal with I_3 and I_4.

In what follows, ν will always represent the outwards pointing normal; in particular, a change of sign may be required for some terms, since the normal ν appearing in the integrals $I_1 \quad I_4$ is $\nu = (x-y)/\varepsilon$ (pointing towards x) on $\partial B_\varepsilon(x)$. Consider I_s :

$$\begin{aligned} I_3 &= \int_{\partial\Omega} \Delta\Gamma(x-\cdot)\partial_\nu u - \frac{1}{2\pi}(\log\varepsilon + 1)\int_{\partial B_\varepsilon(x)} \partial_\nu u \\ &= \int_{\partial\Omega} \Delta\Gamma(x-\cdot)\partial_\nu u - \underbrace{\frac{1}{2\pi}(\log\varepsilon + 1)\int_{B_\varepsilon(x)} \Delta u}_{=:I_3'} \end{aligned}$$

and

$$\left|I_3'\right| \leq \frac{1}{2\pi}|\log\varepsilon + 1| \max_{B_\varepsilon(x)}|\Delta u| \int_{B_\varepsilon(x)} 1 = \frac{\varepsilon^2}{2}|\log\varepsilon + 1| \max_{B_\varepsilon(x)}|\Delta u| \to 0 \qquad (\text{as } \varepsilon \to 0).$$

Now consider I_4 : we employ a first order Taylor expansion $(u(y) = u(x) + \nabla u(\eta(y))\cdot(y-x)$, with $\eta : \partial B_\varepsilon(x) \to B_\varepsilon(x)$) to obtain

$$\begin{aligned}
I_4 &= \int_{\partial\Omega} u\partial_\nu \Delta\Gamma(x-\cdot) - \int_{\partial B_\varepsilon(x)} u\partial_\nu \Delta\Gamma(x-\cdot) \\
&= \int_{\partial\Omega} u\partial_\nu \Delta\Gamma(x-\cdot) - \int_{\partial B_\varepsilon(x)} (u(x) + \nabla u(\eta(y))\cdot(y-x))\partial_\nu \Delta\Gamma(x-y)\mathrm{d}S(y) \\
&= \int_{\partial\Omega} u\partial_\nu \Delta\Gamma(x-\cdot) - u(x)\underbrace{\int_{\partial B_\varepsilon(x)} \partial_\nu \Delta\Gamma(x-\cdot)}_{=1} \\
&\qquad \underbrace{- \int_{\partial B_\varepsilon(x)} \nabla u(\eta(y))\cdot(y-x)\partial_\nu \Delta\Gamma(x-y)\mathrm{d}S(y)}_{=:I_4'}
\end{aligned}$$

and

$$|I_4'| \le \max_{B_\varepsilon(x)} |\nabla u| \varepsilon \int_{\partial B_\varepsilon(x)} |\partial_\nu \Delta\Gamma(x-\cdot)| = \varepsilon \max_{B_\varepsilon(x)} |\nabla u| \to 0 \qquad (\text{as } \varepsilon \to 0).$$

The claim follows.

Exercise 3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary and G be the Green function for the domain Ω.

(i) Fix $x, y \in \Omega$. Write

$$v(z) := G(z,x), \quad w(z) := G(z,y), \qquad \text{for } z \in \Omega.$$

For $0 < \varepsilon < |x-y|/2$, prove

$$\int_{\partial B_\varepsilon(x)} (v\partial_\nu w - w\partial_\nu v) = \int_{\partial B_\varepsilon(y)} (w\partial_\nu v - v\partial_\nu w)$$

where ν denotes the inward unit normal vector on $\partial B_\varepsilon(x) \cup \partial B_\varepsilon(y)$.

(ii) Prove

$$G(y,x) = G(x,y) \qquad \text{for all } x, y \in \Omega.$$

Solution:

(i) We have $\Delta v(z) = 0$ for $z \ne x$, $\Delta w(z) = 0$ for $z \ne y$, and $v|_{\partial\Omega} = 0 = w|_{\partial\Omega}$. For sufficiently small $\varepsilon > 0$, we apply Green's second identity on $\Omega_\varepsilon := \Omega \setminus (B_\varepsilon(x) \cup B_\varepsilon(y))$ for v and w to obtain

$$\int_{\partial\Omega_\varepsilon} (v\partial_\nu w - w\partial_\nu v) = 0$$

Since $v|_{\partial\Omega} = 0 = w|_{\partial\Omega}$, this implies that

$$\int_{\partial B_\varepsilon(x)} (v\partial_\nu w - w\partial_\nu v) = \int_{\partial B_\varepsilon(y)} (w\partial_\nu v - v\partial_\nu w)\,, \tag{1}$$

where ν denotes the inward unit normal vector on $\partial B_\varepsilon(x) \cup \partial B_\varepsilon(y)$.

(ii) We will show $v(y) = w(x)$. We compute the limits of the two terms on both sides of (1) as $\varepsilon \to 0^+$. Since w is smooth near x, we have $|\nabla w| \le M$ on $\overline{B_\varepsilon(x)}$ (provided ε is small enough), and thus

$$\begin{aligned}\left|\int_{\partial B_\varepsilon(x)} v \partial_\nu w\right| &\le M\varepsilon^{n-1} \sup_{z\in\partial B_\varepsilon(x)} |v(z)| \\ &\le M\varepsilon^{n-1}\left(M' + \sup_{z\in\partial B_\varepsilon(x)} |\Gamma(x-z)|\right) \\ &\le M\varepsilon^{n-1}\left(M' + (|\ln\varepsilon|+1)\varepsilon^{2-n}\right) \to 0 \qquad \left(\text{as } \varepsilon \to 0^+\right),\end{aligned}$$

where we used the smoothness of the corrector function $|h(\cdot, x)| \le M'$ on $\overline{B_\varepsilon(x)}$. Moreover, since $h(\cdot, x)$ is smooth in Ω, we have

$$\lim_{\varepsilon\to0^+} \int_{\partial B_\varepsilon(x)} w\partial_\nu v = \lim_{\varepsilon\to0^+} \int_{\partial B_\varepsilon(x)} w\partial_\nu \Gamma(x-\cdot) + 0 = w(x)$$

where for the last equality we recall the proof of the Green representation formula. Hence,

$$\lim_{\varepsilon\to0^+} \text{LHS of (1)} = -w(x)$$

Likewise, $\lim_{\varepsilon\to0^+}$ RHS of (1) $= -v(y)$, and thus $w(x) = v(y)$. This completes the proof.

Exercise 4. Consider the half-plane $\mathbb{R}^2 \supset \Omega = \mathbb{R} \times]0, \infty[$. Given $y = (y_1, y_2) \in \Omega$, we define by reflection $y^\star = (y_1, -y_2)$.

(i) Given the fundamental solution of the Laplace operator $\Phi(x-y) = -(2\pi)^{-1}\log|x-y|$, show that $G(x,y) = \Phi(x-y) - \Phi(x-y^*)$ is the Green function for Ω.

Hint: You need to show that, for any $y \in \Omega$, $h(x,y) := G(x,y) - \Phi(x-y)$ is harmonic in Ω and that $G(\cdot, y) = 0$ on $\partial\Omega$.

(ii) Derive formally the expression of the corresponding Poisson integral

$$u(y) = -\int_{\partial\Omega} \partial_\nu G(x,y) g(x)\, \mathrm{d}S(x), \tag{PI-hp}$$

where $\partial_\nu G(x,y)$ is the normal derivative of $G(\cdot, y)$ at $x \in \partial\Omega$.

(iii) Prove that, if g is continuous and compactly supported, (PI-hp) actually represents a bounded solution of the Dirichlet problem over Ω, that is, $-\Delta u = 0$ in Ω and $\lim_{\substack{y\to y_0,\\ y\in\Omega}} u(y) = g(y_0)$ for each point $y_0 \in \partial\Omega$.

(iv) Prove that (PI-hp) satisfies the following maximum principle:

$$\sup_{\bar\Omega} u = \sup_{\partial\Omega} u.$$

(v) Show that, if g is continuous and compactly supported, $u(y) \to 0$ as $|y| \to \infty$, where u is as in (PI-hp). Moreover, show that u is the unique solution to the problem

$$\begin{cases} -\Delta u(x) = 0, & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega, \end{cases}$$

such that $\lim_{|y|\to\infty} u(y) = 0$.

Solution:

(i) Let $y \in \Omega$. We know $-\Delta\Phi(\cdot - y) = 0$ in $\Omega \backslash \{y\}$. Thus, since $y^\star \notin \Omega$, we have $-\Delta\Phi(\cdot - y^\star) = 0$ in Ω. Moreover, for $x_1 \in \mathbb{R}$, $y \in \Omega$, we have

$$G((x_1, 0), y) = -(2\pi)^{-1} \log \sqrt{(x_1 - y_1)^2 + (-y_2)^2} + (2\pi)^{-1} \log \sqrt{(x_1 - y_1)^2 + (y_2)^2} = 0$$

and thus G is the Green function for Ω.

(ii) The gradient of $G(\cdot, y)$ at x is

$$\left[\frac{-(x_1 - y_1) + (x_1 - y_1)}{2\pi|x - y|^2}, \frac{-(x_2 - y_2) + (x_2 + y_2)}{2\pi|x - y|^2}\right]^\top = \left[0, \frac{y_2}{\pi|x - y|^2}\right]^\top$$

It follows that the normal derivative $\left(\nu = [0, -1]^\top\right)$ of $G(\cdot, y)$ at $x \in \partial\Omega$ is

$$\partial_\nu G(x, y) = -\frac{1}{\pi} \frac{y_2}{|x - y|^2}$$

Thus, formally we have

$$u(y) = -\int_{\partial\Omega} g(x) \partial_\nu G(x, y) \, \mathrm{d}S(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y_2 g(x_1)}{(x_1 - y_1)^2 + y_2^2} \, \mathrm{d}x_1$$

(iii) We first show the boundedness of (PI-hp) on Ω. Let $H(x_1; y) := \frac{1}{\pi} \frac{y_2}{(x_1 - y_1)^2 + y_2^2} \geq 0$ for $x_1 \in \mathbb{R}$, $y \in \Omega$. We have

$$\int_{\mathbb{R}} H(x_1; y) \, \mathrm{d}x_1 = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\mathrm{d}\zeta}{\zeta^2 + 1} = 1 \tag{2}$$

Now, the boundedness of u defined in (PI-hp) on Ω can be checked as

$$u(y) \leq \max_{\mathbb{R}} |g| \int_{\mathbb{R}} H(x_1; y) \, \mathrm{d}x_1 = \max_{\mathbb{R}} |g|. \tag{3}$$

Next, we show that u is harmonic in Ω. First, observe that $H(x_1; \cdot)$ is harmonic in Ω for all $x_1 \in \mathbb{R}$:

$$\frac{\partial^2 H}{\partial y_1^2} = \frac{6(x_1 - y_1)^2 y_2 - 2y_2^3}{\left((x_1 - y_1)^2 + y_2^2\right)^3} \quad \text{and} \quad \frac{\partial^2 H}{\partial y_2^2} = \frac{-6(x_1 - y_1)^2 y_2 + 2y_2^3}{\left((x_1 - y_1)^2 + y_2^2\right)^3}.$$

Since H is smooth in $\mathbb{R} \times \Omega$, and g is bounded and compactly supported, we can exchange integral and derivative(s), e.g. by Lebesgue's dominated convergence theorem, and deduce

$$\Delta u(y) = \int_{\mathbb{R}} g\left(x_1\right) \Delta_y H\left(x_1; y\right) \mathrm{d}x_1 = 0$$

More precisely, to apply Lebesgue's DCT, we consider, for instance, $\partial_{y_2} u$ at $y \in \Omega$: for all $x_1 \in \mathbb{R}$ and $0 < h < y_2/2$,

$$\begin{aligned}
&\left| g\left(x_1\right) \frac{H\left(x_1; \left(y_1, y_2 + h\right)\right) - H\left(x_1; \left(y_1, y_2\right)\right)}{h} \right| \leq \max_{\eta \in [y_2/2, 3y_2/2]} \left| g\left(x_1\right) \frac{\partial H}{\partial y_2}\left(x_1; \left(y_1, \eta\right)\right) \right| \\
&\leq \max_{\mathbb{R}} |g| \max_{\substack{\zeta \in \operatorname{supp} g \\ \eta \in [y2/2, 3y2/2]}} \left| \frac{\partial H}{\partial y_2}\left(\zeta; \left(y_1, \eta\right)\right) \right| \mathbb{1}_{\operatorname{supp}(g)}\left(x_1\right) \\
&=: \varphi\left(x_1\right).
\end{aligned}$$

and note that the function φ is integrable on $\mathbb{R}$ independently of h.

Now, we will check the boundary condition. We will show that $\lim_{\Omega \ni y \to (0,0)} u(y) = g(0)$. The argument is analogous for $y_1 \neq 0$. Let $\varepsilon > 0$ be fixed, and choose $\delta = \delta(\varepsilon)$ small enough that

$$\left|x_1\right| < \delta \Longrightarrow \left|g\left(x_1\right) - g(0)\right| < \varepsilon/2$$

Thanks to (2),

$$|u(y) - g(0)| \leq \int_{-\delta}^{\delta} H\left(x_1; y\right) \left|g\left(x_1\right) - g(0)\right| \mathrm{d}x_1 + \int_{\mathbb{R} \setminus [-\delta, \delta]} H\left(x_1; y\right) \left|g\left(x_1\right) - g(0)\right| \mathrm{d}x_1$$

The first integral is obviously not larger than $\varepsilon/2$. For the second, we have

$$\begin{aligned}
\int_{\mathbb{R} \setminus [-\delta, \delta]} H\left(x_1; y\right) \left|g\left(x_1\right) - g(0)\right| \mathrm{d}x_1 &\leq 2 \max_{\mathbb{R}} |g| \int_{\mathbb{R} \setminus [-\delta, \delta]} H\left(x_1; y\right) \mathrm{d}x_1 \\
&\leq \frac{2y_2}{\pi} \max_{\mathbb{R}} |g| \int_{\mathbb{R} \setminus [-\delta, \delta]} \frac{1}{\left(x_1 - y_1\right)^2} \mathrm{d}x_1
\end{aligned}$$

Now, assume $\left|y_1\right| < \delta/2$. Then, for all $\left|x_1\right| > \delta, \left(x_1 - y_1\right)^2 \geq x_1^2/4$, and

$$\int_{\mathbb{R} \setminus [-\delta, \delta]} H\left(x_1; y\right) \left|g\left(x_1\right) - g(0)\right| \mathrm{d}x_1 \leq \frac{2y_2}{\pi} \max_{\mathbb{R}} |g| \int_{\mathbb{R} \setminus [-\delta, \delta]} \frac{4}{x_1^2} \mathrm{d}x_1 = \frac{2y_2}{\pi} \max_{\mathbb{R}} |g| \frac{8}{\delta},$$

which is smaller than $\varepsilon/2$, provided $y_2 < (\pi\varepsilon\delta)/\left(16 \max_{\mathbb{R}} |g|\right)$. Hence, we have the following implication:

$$y \in \Omega, |y| < \min\left\{\frac{\delta}{2}, \frac{\pi\varepsilon\delta}{16 \max_{\mathbb{R}} |g|}\right\} \Longrightarrow |u(y) - g(0)| < \varepsilon.$$

Finally, we conclude that the boundary condition $u(\cdot, 0) = g$ holds, and, together with (3), that u is bounded on $\bar{\Omega}$.

(iv) We show the maximum principle for an arbitrary $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that u is harmonic in Ω and $\sup_{\bar{\Omega}} |u| < \infty$. For $(y_1, y_2) \in \bar{\Omega}$, let

$$\varphi(y_1, y_2) := \log \sqrt{y_1^2 + (y_2 + 1)^2} \geq 0$$

Then, $\varphi \in C(\bar{\Omega}), \Delta\varphi = 0$ in Ω, and $\varphi(0,0) = 0$. Let

$$v_\varepsilon := u - \varepsilon\varphi$$

Then, v_ε is harmonic in Ω and continuous on $\bar{\Omega}$. Thus, in view of the maximum principle for any $\Omega_r := B(0,r) \cap \Omega$ we have

$$\max_{\overline{\Omega_r}} v_\varepsilon = \max_{\partial\Omega_r} v_\varepsilon$$

Now, $\partial\Omega_r$ consist of the straight part and curved part: $\partial\Omega_r = \left(\overline{B(0,r)} \cap \{y_2 = 0\}\right) \cup (\partial B(0,r) \cap \Omega\ )$.

We already know that $\sup_{\partial\Omega} u \leq \sup_{\bar{\Omega}} |u| \leq M$. On the curved part $\partial B(0,r) \cap \Omega$, we have

$$\varphi(y_1, y_2) \geq \log r \quad \text{for} \quad (y_1, y_2) \in \partial B(0,r) \cap \Omega$$

and there exists $R = R(\varepsilon) > 0$ such that

$$v_\varepsilon \leq M - \varepsilon\varphi \leq \sup_{\partial\Omega} u \quad \text{on } \partial B(0,r) \cap \Omega \quad \text{for all } r > R$$

On the other hand, on the straight part we have $v_\varepsilon \leq u \leq \sup_{\partial\Omega} u$. In either case we have

$$\max_{\overline{\Omega_r}} v_\varepsilon = \max_{\partial\Omega_r} v_\varepsilon \leq \sup_{\partial\Omega} u$$

We now take $r \to \infty$ to find $\sup_{\bar{\Omega}} v_\varepsilon \leq \sup_{\partial\Omega} u$. But from $u = v_\varepsilon + \varepsilon\varphi$, this implies

$$u(y_1, y_2) \leq \sup_{\partial\Omega} u + \varepsilon\varphi(y_1, y_2) \quad \text{for any } (y_1, y_2) \in \bar{\Omega}$$

Since the left hand side is independent of ε, we take $\varepsilon \to 0$ to obtain $\sup_{\bar{\Omega}} u \leq \sup_{\partial\Omega} u$.

(v) Since $\max\{|y_1|, |y_2|\} \leq |y| \leq \sqrt{2}\max\{|y_1|, |y_2|\}$, it is equivalent to show $|u(y)| \to 0$ as $\max\{|y_1|, |y_2|\} \to 0$. Let R be such that $\operatorname{supp}(g) \subset B_R(0)$. Take $M > 0$ such that $R \leq M/2$. Suppose $\max\{|y_1|, |y_2|\} > M$. If $\max\{|y_1|, |y_2|\} = |y_2|$, then,

$$|u(y)| \leq \frac{1}{\pi}\int_{\mathbb{R}} \left|\frac{y_2 g(x_1)}{(x_1 - y_1)^2 + y_2^2}\right| \mathrm{d}x_1 \leq \frac{1}{\pi}\int_{B_R(0)} \left|\frac{y_2 g(x_1)}{y_2^2}\right| \mathrm{d}x_1 \leq \frac{1}{\pi M}|B_R(0)| \max_{\mathbb{R}} |g|.$$

If $\max\{|y_1|, |y_2|\} = |y_1|$, from $|y_1| > M, |y_1| \geq |y_2|$, and $M/2 \geq R$, we have

$$\begin{aligned}|u(y)| &\leq \frac{1}{\pi}\int_{B_R(0)} \left|\frac{y_2 g(x_1)}{(x_1 - y_1)^2 + y_2^2}\right| \mathrm{d}x_1 \leq \frac{1}{\pi}\int_{B_R(0)} \frac{M|g(x_1)|}{(R - |y_1|)^2 + y_2^2} \mathrm{d}x_1 \\ &< \frac{1}{\pi}\int_{B_R(0)} \frac{M|g(x_1)|}{(M-R)^2} \mathrm{d}x_1 \leq \frac{4}{\pi M}|B_R(0)| \max_{\mathbb{R}} |g|.\end{aligned}$$

Hence, we conclude $|u(y)| \to 0$ as $|y| \to \infty$.

To show uniqueness, let u and w be two such solutions. Then, $u - w$ is harmonic in Ω, continuous on $\bar{\Omega}$, and $\lim_{|y|\to\infty}(u(y) - w(y)) = 0$. Hence, following the same argument as in the proof of the comparison principle in the lecture notes, we have $\sup_{\bar{\Omega}}(u - w) = \sup_{\partial\Omega}(u - w)$, which is zero as we have $(u - w)|_{\partial\Omega} = 0$. Similarly, $\sup_{\bar{\Omega}}(w - u) = 0$.