Exercise Sheet 5

Introduction to Partial Differential Equations (W. S. 2024/25)

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- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

Exercise 1. Prove the following result, in which the fundamental solution Φ of the Laplace equation is used to derive a representation formula for the point value of a $C^2(\Omega)$ function in terms of its Laplacian and boundary values.

> *Green's representation formula*: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary. For any $u \in C^2(\bar{\Omega})$ and $y \in \Omega$, we have
>
> $$u(y) = \int_\Omega \Phi(x-y)(-\Delta u(x))\,\mathrm{d}x - \int_{\partial\Omega} \partial_\nu \Phi(x-y) u(x)\,\mathrm{d}S(x) + \int_{\partial\Omega} \Phi(x-y)\partial_\nu u(x)\,\mathrm{d}S(x). \tag{GRF}$$

Context and hints for the proof: The starting point of the proof is Green's integration by parts identity,

$$\int_\Omega (v\Delta u - u\Delta v)\,\mathrm{d}x = \int_{\partial\Omega} (v\partial_\nu u - u\partial_\nu v)\,\mathrm{d}S. \tag{GI}$$

which holds for every $u, v \in C^2(\bar{\Omega})$ and a bounded domain with C^1 boundary. Now fix $y \in \Omega$. To prove the (GRF), the idea is to take $v = \Phi(\cdot - y)$ in (GI). However, $x \mapsto \Phi(x-y)$ is not $C^2(\bar{\Omega})$, and in particular, its Laplacian is not defined in y. So we need to be more careful: to circumvent this problem, we write the identity on the domain $\Omega_\epsilon = \Omega \setminus B_\epsilon(y)$, with ϵ small enough so that $B_\epsilon(y) \subset \Omega$, and let $\epsilon \to 0$.

Solution: Exploiting the fact that $\Phi(x-y) \in C^2(\bar{\Omega}_\epsilon)$ and $\Delta_x \Phi(x-y) = 0$ in Ω_ϵ, we have

$$\underbrace{\int_{\Omega_\epsilon} \Delta u(x)\Phi(x-y)\,\mathrm{d}x}_{C} - \int_{\partial\Omega} \left(\partial_\nu u(x)\Phi(x-y) - u(x)\partial_\nu \Phi(x-y)\right)\,\mathrm{d}S(x)$$

$$= \underbrace{\int_{\partial B_\epsilon(y)} \partial_\nu u(x)\Phi(x-y)\,\mathrm{d}S(x)}_{A} - \underbrace{\int_{\partial B_\epsilon(y)} u(x)\partial_\nu \Phi(x-y)\,\mathrm{d}S(x)}_{B}.$$

We show now that $A \to 0$, $B \to u(y)$, and $C \to \int_\Omega \Delta u(x)\Phi(x-y)\,\mathrm{d}x$ as $\epsilon \to 0$.

For A, we estimate

$$|A| = |\tilde{\Phi}(\epsilon)| \left| \int_{\partial B_\epsilon(y)} \partial_\nu u \, \mathrm{d}S \right| \leq |\tilde{\Phi}(\epsilon)| \epsilon^{n-1} \omega_n \sup_{B_\epsilon(y)} |\nabla u| \longrightarrow 0 \text{ as } \epsilon \to 0.$$

For B, we estimate the difference $|B - u(y)|$:

$$\begin{aligned}
|B - u(y)| &= \left| \int_{\partial B_\epsilon(y)} (u(y) + \nabla u(y) + \theta_x(x-y) \cdot (x-y)) \, \partial_\nu \Phi(x-y) \, \mathrm{d}S(x) - u(y) \right| \\
&\leq \underbrace{\left| u(y) \left(\int_{\partial B_\epsilon(y)} \partial_\nu \Phi(x-y) \, \mathrm{d}S(x) - 1 \right) \right|}_{=0} \\
&\quad + \left| \int_{\partial B_\epsilon(y)} \nabla u(y + \theta_x(x-y)) \cdot (x-y) \partial_\nu \Phi(x-y) \, \mathrm{d}S(x) \right| \\
&\leq \sup_{B_\epsilon(y)} |\nabla u| \epsilon \int_{\partial B_\epsilon(y)} |\partial_\nu \Phi(x-y)| \, \mathrm{d}S(x) \longrightarrow 0 \text{ as } \epsilon \to 0..
\end{aligned}$$

Notice that, in the second line, the fact that $\int_{\partial B_\epsilon(y)} \partial_\nu \Phi(x-y) \, \mathrm{d}S(x) = 1$ is due to ν being the normal outgoing vector to the domain $\Omega \setminus B_\epsilon(y)$, hence the normal *ingoing* vector to $B_\epsilon(y)$. Finally, for the term C, since $\Phi(x-y)$ is integrable in Ω for any $y \in \Omega$, we have

$$\begin{aligned}
\left| C - \int_\Omega \Delta u(x) \Phi(x-y) \, \mathrm{d}x \right| &= \left| \int_{B_\epsilon(y)} \Delta u(x) \Phi(x-y) \, \mathrm{d}x \right| \leq \sup_{B_\epsilon(y)} |\Delta u| \int_{B_\epsilon(y)} |\Phi(x-y)| \, \mathrm{d}x \\
&\leq \sup_{B_\epsilon(y)} |\Delta u| \int_0^\epsilon s^{n-1} \omega_n |\Phi(s)| \, \mathrm{d}s \longrightarrow 0 \text{ as } \epsilon \to 0.
\end{aligned}$$

Exercise 2. Let $u \in C^2(\bar{\Omega})$ be a solution, provided it exists, of the Neumann boundary value problem

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ \partial_\nu u = h, & x \in \partial\Omega, \end{cases}$$

for a smooth domain $\Omega \subset \mathbb{R}^n$ and smooth data f, h which satisfy the compatibility condition

$$\int_\Omega f = -\int_{\partial\Omega} h$$

(i) Fix $y \in \Omega$. Show that the Neumann problem

$$\begin{cases} -\Delta \psi(\cdot, y) = 0, & x \in \Omega, \\ \partial_\nu \psi(\cdot, y) = \partial_\nu \Phi(\cdot - y), & x \in \partial\Omega, \end{cases}$$

has no solution $\psi \in C^2(\bar{\Omega})$.

(ii) Suppose further that $\psi(\cdot, y) \in C^2(\bar{\Omega})$ satisfies

$$\begin{cases} -\Delta\psi(\cdot, y) = 0, & x \in \Omega, \\ \partial_\nu \psi(\cdot, y) = \partial_\nu \Phi(\cdot - y) + \frac{1}{|\partial\Omega|}, & x \in \partial\Omega, \end{cases}$$

exists, where $y \in \Omega$ is fixed arbitrarily. Show that we have

$$u(y) - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} u(x)\,\mathrm{d}S(x) = \int_\Omega f(x)N(x,y)\,\mathrm{d}x + \int_{\partial\Omega} h(x)N(x,y)\,\mathrm{d}S(x),$$

where $N(x,y) \coloneqq \Phi(x-y) - \psi(x,y)$.

Solution:

(i) Using the Green's representation formula for the constant function 1 yields

$$1 = -\int_\Omega \Phi(\cdot - y)\Delta 1 - \int_{\partial\Omega} \partial_\nu \Phi(\cdot - y) + \int_{\partial\Omega} \Phi(\cdot - y)\partial_\nu 1 = 0 - \int_{\partial\Omega} \partial_\nu \Phi(\cdot - y) + 0$$

Now suppose that $\tilde{\psi} \in C^2(\bar{\Omega})$ is a solution to the Neumann problem. Integration by parts yields

$$\int_{\partial\Omega} \partial_\nu \tilde{\psi}(\cdot, y) = 0,$$

which contradicts with $\int_{\partial\Omega} \partial_\nu \tilde{\psi}(\cdot, y) = \int_{\partial\Omega} \partial_\nu \Phi(\cdot - y) = -1$, hence the problem above has no solution.

(ii) Integrating by parts yields

$$\begin{aligned} 0 + \int_\Omega \psi(\cdot, y) f &= \int_\Omega (u\Delta\psi(\cdot, y) - \psi(\cdot, y)\Delta u) \\ &= \int_{\partial\Omega} (u\partial_\nu \psi(\cdot, y) - \psi(\cdot, y)\partial_\nu u) \\ &= \int_{\partial\Omega} \left(u \left(\partial_\nu \Phi(\cdot - y) + \frac{1}{|\partial\Omega|} \right) - \psi(\cdot, y)h \right) \end{aligned}$$

Now, from the Green's representation formula, the solution u of the Neumann problem, if it exists, admits the representation

$$\begin{aligned} u(y) &= -\int_\Omega \Phi(\cdot - y)\Delta u - \int_{\partial\Omega} u\partial_\nu \Phi(\cdot - y) + \int_{\partial\Omega} \Phi(\cdot - y)\partial_\nu u \\ &= \int_\Omega \Phi(\cdot - y) f - \int_{\partial\Omega} u\partial_\nu \Phi(\cdot - y) + \int_{\partial\Omega} \Phi(\cdot - y)h \end{aligned}$$

Adding the above two identities yields

$$u(y) - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} u = \int_\Omega N(\cdot, y) f + \int_{\partial\Omega} N(\cdot, y)h,$$

which is the desired identity.

Exercise 3. The aim of this exercise is to define basic operations over distributions in $\mathcal{D}'(\mathbb{R})$. As has been done for the derivative of a distribution, basic mathematical operations can be defined following the same line of reasoning.

First of all, we introduce the notation used throughout the exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function, we then define the following:

(a) **Translation** of f by $a \in \mathbb{R}$:

$$\tau_a f(x) = f(x+a) \quad \text{for all } x \in \mathbb{R}.$$

(b) **Dilation** of f by $a > 0$:

$$D_a f(x) = f(ax) \quad \text{for all } x \in \mathbb{R}.$$

(c) **Reflection** of f:

$$\tilde{f}(x) = f(-x) \quad \text{for all } x \in \mathbb{R}.$$

(d) **Multiplication** by a function $g \in C^\infty(\mathbb{R})$:

$$(fg)(x) = f(x)g(x).$$

We now extend these definitions to elements of the space $\mathcal{D}'(\mathbb{R})$: Given $T \in \mathcal{D}'(\mathbb{R})$, we can define:

(a) The **translation** by $a \in \mathbb{R}$ of T is the distribution $\tau_a T \in \mathcal{D}'(\mathbb{R})$ such that:

$$\langle \tau_a T, \phi \rangle = \langle T, \tau_{-a}\phi \rangle \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}).$$

(b) The **dilation** by $a > 0$ of T is the distribution $D_a T \in \mathcal{D}'(\mathbb{R})$ such that:

$$\langle D_a T, \phi \rangle = \left\langle T, \frac{1}{a} D_{1/a}\phi \right\rangle \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}).$$

(c) The **reflection** of T is the distribution $\tilde{T} \in \mathcal{D}'(\mathbb{R})$ such that:

$$\left\langle \tilde{T}, \phi \right\rangle = \left\langle T, \tilde{\phi} \right\rangle \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}).$$

(d) The **multiplication** of T by $g \in C^\infty(\mathbb{R})$ is a distribution $gT \in \mathcal{D}'(\mathbb{R})$ such that:

$$\langle gT, \phi \rangle = \langle T, g\phi \rangle \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}).$$

Given $f \in L^1_{\text{loc}}(\mathbb{R})$, we can associate to f the distribution $T_f \in \mathcal{D}'(\mathbb{R})$ such that:

$$\langle T_f, \phi \rangle = \int_{\mathbb{R}} f(x)\phi(x)\, dx \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}).$$

Prove that for T_f built in such a way, the previous definitions hold true.

Based on the previous computations, one can extend these definitions to a generic $T \in \mathcal{D}'(\mathbb{R})$. Starting from these definitions, prove that:

(a) $(\tau_a T)' = \tau_a(T')$.

(b) $(D_a T)' = a\, D_a(T')$.

(c) $(\tilde{T})' = -\tilde{T'}$.

(d) Given $T \in \mathcal{D}'(\mathbb{R})$, $g \in C^\infty(\mathbb{R})$, prove that $(gT)' = g'T + gT'$.

We say that a distribution is **even** if $\tilde{T} = T$ in $\mathcal{D}'(\mathbb{R})$, similarly a distribution is **odd** if $\tilde{T} = -T$ in $\mathcal{D}'(\mathbb{R})$. Prove that:

(a) δ_0 is an even distribution.

(b) δ_0' is an odd distribution.

Prove that the following identities hold in $\mathcal{D}'(\mathbb{R})$:

(a) $D_a \tau_{-2} \delta_0 = \frac{1}{3}\delta_3$.

(b) $(D_2\,(e^{-x}\delta_0'))' = \frac{1}{2}\delta_0' + \frac{1}{4}\delta_0''$.

Solution: Let $f \in L^1_{\text{loc}}(\mathbb{R})$, and consider the distribution $T_f \in \mathcal{D}'(\mathbb{R})$ defined by:

$$\langle T_f, \phi \rangle = \int_{\mathbb{R}} f(x)\phi(x)\, dx \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}).$$

We will show that the operations defined for distributions coincide with those for functions when T_f is associated to f.

(a) **Translation of T_f:**

$$\begin{aligned}\langle \tau_a T_f, \phi \rangle = \langle T_f, \tau_{-a}\phi \rangle &= \int_{\mathbb{R}} f(x)\phi(x-a)\, dx \\ &= \int_{\mathbb{R}} f(y+a)\phi(y)\, dy = \int_{\mathbb{R}} \tau_a f(y)\phi(y)\, dy = \langle T_{\tau_a f}, \phi \rangle .\end{aligned}$$

Therefore, $\tau_a T_f = T_{\tau_a f}$.

(b) **Dilation of T_f:**

$$\begin{aligned}\langle D_a T_f, \phi \rangle = \left\langle T_f, \frac{1}{a} D_{1/a}\phi \right\rangle &= \frac{1}{a}\int_{\mathbb{R}} f(x)\phi\left(\frac{x}{a}\right)\, dx \\ &= \int_{\mathbb{R}} f(ay)\phi(y)\, dy = \langle T_{D_a f}, \phi \rangle .\end{aligned}$$

Therefore, $D_a T_f = T_{D_a f}$.

(c) **Reflection of T_f:**

$$\begin{aligned}\left\langle \tilde{T}_f, \phi \right\rangle = \left\langle T_f, \tilde{\phi} \right\rangle &= \int_{\mathbb{R}} f(x)\phi(-x)\, dx \\ &= \int_{\mathbb{R}} f(-y)\phi(y)\, dy = \left\langle T_{\tilde{f}}, \phi \right\rangle .\end{aligned}$$

Therefore, $\tilde{T}_f = T_{\tilde{f}}$.

(d) **Multiplication by g:**

$$\langle gT_f, \phi \rangle = \langle T_f, g\phi \rangle = \int_{\mathbb{R}} f(x)g(x)\phi(x)\,dx = \langle T_{fg}, \phi \rangle\,.$$

Therefore, $gT_f = T_{fg}$.

This shows that for T_f, the operations defined for distributions correspond to those for functions.

Next, we will prove the derivation rules for distributions.

(a) **Proof that $(\tau_a T)' = \tau_a(T')$:**

For all $\phi \in \mathcal{D}(\mathbb{R})$,

$$\begin{aligned}\left\langle (\tau_a T)', \phi \right\rangle &= -\left\langle \tau_a T, \phi' \right\rangle = -\left\langle T, \tau_{-a}\phi' \right\rangle \\ &= -\left\langle T, (\tau_{-a}\phi)' \right\rangle \quad \text{since } \tau_{-a}\phi' = (\tau_{-a}\phi)' \\ &= \left\langle T', \tau_{-a}\phi \right\rangle = \left\langle \tau_a T', \phi \right\rangle.\end{aligned}$$

Therefore, $(\tau_a T)' = \tau_a(T')$.

(b) **Proof that $(D_a T)' = a\,D_a(T')$:**

For all $\phi \in \mathcal{D}(\mathbb{R})$,

$$\begin{aligned}\left\langle (D_a T)', \phi \right\rangle &= -\left\langle D_a T, \phi' \right\rangle = -\left\langle T, \frac{1}{a} D_{1/a}\phi' \right\rangle \\ &= -\left\langle T, \left(D_{1/a}\phi\right)' \right\rangle \quad \text{since } \frac{1}{a} D_{1/a}\phi' = \left(D_{1/a}\phi\right)' \\ &= \left\langle T', D_{1/a}\phi \right\rangle = \left\langle D_a T', \phi \right\rangle a.\end{aligned}$$

Therefore, $(D_a T)' = a\,D_a(T')$.

(c) **Proof that $(\tilde{T})' = -\tilde{T}'$:**

For all $\phi \in \mathcal{D}(\mathbb{R})$,

$$\begin{aligned}\left\langle (\tilde{T})', \phi \right\rangle &= -\left\langle \tilde{T}, \phi' \right\rangle = -\left\langle T, \tilde{\phi}' \right\rangle \\ &= -\left\langle T, \left(\tilde{\phi}\right)' \right\rangle \quad \text{since } \tilde{\phi}' = \left(\tilde{\phi}\right)' \\ &= \left\langle T', \tilde{\phi} \right\rangle = \left\langle -\tilde{T}', \phi \right\rangle.\end{aligned}$$

Therefore, $(\tilde{T})' = -\tilde{T}'$.

(d) **Proof that $(gT)' = g'T + gT'$:**

For all $\phi \in \mathcal{D}(\mathbb{R})$,

$$\begin{aligned}\left\langle (gT)', \phi \right\rangle &= -\left\langle gT, \phi' \right\rangle = -\left\langle T, g\phi' \right\rangle \\ &= -\left\langle T, (g\phi)' - g'\phi \right\rangle = -\left(\left\langle T, (g\phi)' \right\rangle - \left\langle T, g'\phi \right\rangle\right) \\ &= -\left(-\left\langle T', g\phi \right\rangle - \left\langle T, g'\phi \right\rangle\right) \\ &= \left\langle T', g\phi \right\rangle + \left\langle T, g'\phi \right\rangle = \left\langle gT', \phi \right\rangle + \left\langle g'T, \phi \right\rangle.\end{aligned}$$

Therefore, $(gT)' = g'T + gT'$.

Now, we will prove that δ_0 is even and δ_0' is odd.

(a) **Proof that δ_0 is even:**

For all $\phi \in \mathcal{D}(\mathbb{R})$,

$$\left\langle \tilde{\delta}_0, \phi \right\rangle = \left\langle \delta_0, \tilde{\phi} \right\rangle = \tilde{\phi}(0) = \phi(-0) = \phi(0) = \langle \delta_0, \phi \rangle .$$

Therefore, $\tilde{\delta}_0 = \delta_0$, so δ_0 is even.

(b) **Proof that δ_0' is odd:**

For all $\phi \in \mathcal{D}(\mathbb{R})$,

$$\begin{aligned} \left\langle \tilde{\delta}_0', \phi \right\rangle &= \left\langle \delta_0', \tilde{\phi} \right\rangle = -\left\langle \delta_0, (\tilde{\phi})' \right\rangle = -(\tilde{\phi})'(0) \\ &= -\left(\frac{d}{dx}\phi(-x)\bigg|_{x=0} \right) = -(-\phi'(0)) = \phi'(0) = \left\langle -\delta_0', \phi \right\rangle . \end{aligned}$$

Therefore, $\tilde{\delta}_0' = -\delta_0'$, so δ_0' is odd.