Exercise Sheet 4

Introduction to Partial Differential Equations (W. S. 2024/25)

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- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

Exercise 1. Given a function $u \in C^2(\mathbb{R}^n)$, prove Bochner's identity:[1]

$$\Delta \frac{1}{2} |\nabla u|^2 = \nabla u \cdot \nabla \Delta u + |D^2 u|^2.$$

(In particular, if u is harmonic, then $\Delta \frac{1}{2} |\nabla u|^2 = |D^2 u|^2$.) Use it to prove that an harmonic function u on $\mathbb{R}^n$, with $\int_{\mathbb{R}^n} |\nabla u|^2 \, \mathrm{d}x < \infty$, is constant.

Hint: Consider the quantity $\frac{1}{2} \int_{\mathbb{R}^n} \varphi_R \Delta |\nabla u|^2 \, \mathrm{d}x$, where $\varphi_R(x) := \varphi(x/R)$ and φ is a non-negative smooth function φ such that

$$\varphi(x) = \begin{cases} 1 & \text{in } B_1(0) \\ 0 & \text{in } \mathbb{R}^n \setminus B_2(0) \end{cases}$$

and $|\Delta \varphi| \leq M$ for some $M > 0$.

Solution: The first part is a computation.

Let us prove the second part (Liouville-type theorem). Fix a non-negative smooth function φ such that

$$\varphi(x) = \begin{cases} 1 & \text{in } B_1(0) \\ 0 & \text{in } \mathbb{R}^n \setminus B_2(0) \end{cases}$$

and $|\Delta \varphi| \leq M$ for some $M > 0$. Then define the rescaled function $\varphi_R(x) := \varphi(x/R)$. We have $|\Delta \varphi_R(x)| \leq M/R^2$.

By the Bochner's identity,

$$\Delta \frac{1}{2} |\nabla u|^2 = \nabla u \cdot \nabla \Delta u + |D^2 u|^2 = |D^2 u|^2,$$

we have

$$\int_{\mathbb{R}^n} \varphi_R |D^2 u|^2 \, \mathrm{d}x = \frac{1}{2} \int_{\mathbb{R}^n} \varphi_R \Delta |\nabla u|^2 \, \mathrm{d}x.$$

[1] Named after Salomon Bochner.

Using the properties of φ and integration by parts, we then deduce

$$\int_{B_R} |D^2u|^2 \,\mathrm{d}x \le \int_{\mathbb{R}^n} \varphi_R |D^2u|^2 \,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^n} \varphi_R \Delta |\nabla u|^2 \,\mathrm{d}x \le \frac{M}{2R^2}\int_{B_{2R}} |\nabla u|^2 \,\mathrm{d}x \le \frac{CM}{2R^2}.$$

Letting $R \to \infty$, we deduce $\int_{\mathbb{R}^n} |D^2u|^2 \,\mathrm{d}x = 0$, i.e., $D^2u \equiv 0$. This yields that u is affine: $u(x) = a \cdot x + b$. But the condition $\int_{\mathbb{R}^n} |\nabla u|^2 \,\mathrm{d}x < \infty$ implies $a = 0$, thus u is actually constant.

Exercise 2. Consider a sequence of non-negative harmonic functions $\{u_i\}_{i\in\mathbb{N}}$, defined on a domain Ω in $\mathbb{R}^N$. Use Harnack's inequality to show that, if $\sum_{i=0}^{\infty} u_i$ converges at some $x_0 \in \Omega$, then it converges uniformly on any compact set $K \subset \Omega$. Deduce that the sum U of the series is non-negative and harmonic everywhere on Ω.

Solution: We wish to prove uniform convergence on any given compact $K \subset \Omega$. Since Ω is connected, we may assume K is connected and that it contains x_0 (if not, just choose another compact set containing the original one). By Harnack's inequality

$$\max_K u_i \le \gamma_i \min_K u_i \le \gamma_i u_i(x_0).$$

Then

$$\sum_{i=0}^{\infty} \max_K u_i(x) \le \sum_{i=0}^{\infty} \gamma_i u_i(x_0) < \infty,$$

and, by the Weierstrass criterion, the series $\sum_{i=0}^{\infty} u_i(x)$ converges uniformly on K. But $K \subset \Omega$ is arbitrary, so the series converges at any point in Ω, implying that the sum U is defined on Ω, and is non-negative as a sum of non-negative terms. Also, the u_i are continuous on Ω, hence uniform convergence on compact subsets of Ω guarantees that U is continuous on Ω.

To show U is harmonic, then, it suffices to show it satisfies the mean-value property. For any $B_r(x) \subset\subset \Omega$,

$$\begin{aligned}
\frac{1}{|B_r(x)|}\int_{B_r(x)} U(y)\,\mathrm{d}y &= \frac{1}{|B_r(x)|}\int_{B_r(x)} \left(\sum_{i=0}^{\infty} u_i(y)\right) \mathrm{d}y \\
&= \sum_{i=0}^{\infty} \frac{1}{|B_r(x)|}\int_{B_r(x)} u_i(y)\,\mathrm{d}y \\
&= \sum_{i=0}^{\infty} u_i(x) = U(x),
\end{aligned}$$

and the claim follows (we rely on uniform convergence in order to swap sum and integral).

Exercise 3. Compute the limit in $\mathcal{D}'(\mathbb{R})$ of the following sequences of distributions.

(i) T_{f_n}, where $f_n(x) = \frac{n}{2}\chi_{[-\frac{1}{n},\frac{1}{n}]}$.

(ii) T_{g_n}, where $g_n(x) = \sin(nx)$.

(iii) $D_n = n\left(\delta_{\frac{1}{n}} - \delta_0\right)$.

(iv) T_{f_n}, where $f_n(x) = \frac{1}{|x|+\frac{1}{n}}$.

Solution: Recall that $T_n \to T$ in $\mathcal{D}'(\Omega)$ if $\langle T_n, \varphi\rangle \to \langle T, \varphi\rangle$ for every $\varphi \in \mathcal{D}(\Omega)$.

(i) For every $\varphi \in \mathcal{D}(\Omega)$,

$$\langle T_{f_n}, \varphi\rangle = \int_{-\infty}^{\infty} f_n(x)\varphi(x)\,\mathrm{d}x = \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} \varphi(x)\,\mathrm{d}x \to \varphi(0)$$

for the mean value theorem, since $\varphi \in C_c^\infty(\mathbb{R})$. Hence, $T_n \to \delta_0$.

(ii)

$$\langle T_{g_n}, \varphi\rangle = \int_{-\infty}^{\infty} \sin(nx)\varphi(x)\,\mathrm{d}x = -\frac{1}{n}\cos(nx)\varphi(x)\Big|_{-\infty}^{\infty} + \frac{1}{n}\int_{-\infty}^{\infty}\cos(nx)\varphi'(x)\,\mathrm{d}x.$$

The first term is equal to zero as $\varphi \in \mathcal{D}$ and thus it has compact support. The second term can be bounded as

$$\frac{1}{n}\int_{-\infty}^{\infty}\cos(nx)\varphi'(x)\,\mathrm{d}x \le \frac{1}{n}\int_{-\infty}^{\infty}|\varphi'(x)|\,\mathrm{d}x \le \frac{C}{n}.$$

Therefore, we reach the surprising conclusion that $T_{g_n} \to 0$ in $\mathcal{D}'$.

(iii)

$$\langle D_n, \varphi\rangle = \frac{\varphi\left(\frac{1}{n}\right) - \varphi(0)}{\frac{1}{n}} \to \varphi'(0) = \langle \delta_0', \varphi\rangle = -\langle \delta_0', \varphi\rangle.$$

(iv)

$$\langle T_n, \varphi\rangle = \int_{\mathbb{R}} \frac{1}{|x|+\frac{1}{n}}\varphi(x)\,\mathrm{d}x$$

and this integral diverges for $n \to \infty$ and for every $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi(0) \neq 0$ and $\varphi \ge 0$ in $\mathbb{R}$. Indeed, for such a φ there exists $\delta, c > 0$ so that

$$\int_{\mathbb{R}} \frac{1}{|x|+\frac{1}{n}}\varphi(x)\,\mathrm{d}x \ge \int_{-\delta}^{\delta} \frac{1}{|x|+\frac{1}{n}}\varphi(x)\,\mathrm{d}x \ge c\int_{-\delta}^{\delta}\frac{\mathrm{d}x}{|x|+\frac{1}{n}} = 2c\left[\log\left(\delta + \frac{1}{n}\right) + \log(n)\right] \to \infty.$$

Hence, the sequence does not converge to a distribution in $\mathcal{D}'(\mathbb{R})$.

Exercise 4. Compute, using the definition, the distributional derivative of the following distributions in $\mathcal{D}'(\mathbb{R})$.

(i) $T = |x|$.

(ii) $T = H(x)$, where $H(x) = \chi_{[0,+\infty)}$ is the Heaviside function.

(iii) $T = \chi_{[-a,a]}$ for some $a > 0$.

Solution: Recall that the distributional derivative of an element $T \in \mathcal{D}'(\mathbb{R})$ is a distribution $T' \in \mathcal{D}'(\mathbb{R})$ such that

$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}).$$

(i) $T = |x|$:

$$\begin{aligned}\langle T_{|x|}, \varphi \rangle = -\langle |x|, \varphi' \rangle = \int_{\mathbb{R}} |x| \varphi'(x) \, \mathrm{d}x &= -\int_0^{\infty} x\varphi'(x) \, \mathrm{d}x - \int_{-\infty}^{0} x\varphi'(x) \, \mathrm{d}x \\ &= -\left[-x\varphi(x)\right]_{-\infty}^{0} - \int_{-\infty}^{0} \varphi(x) \, \mathrm{d}x - \int_0^{\infty} \varphi(x) \, \mathrm{d}x \\ &= -\int_0^{\infty} \varphi(x) \, \mathrm{d}x + \int_{-\infty}^{0} \varphi(x) \, \mathrm{d}x = \int_{\mathbb{R}} \operatorname{sign}(x)\varphi(x) \, \mathrm{d}x \\ &= \langle T_{\operatorname{sign}(x)}, \varphi \rangle \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}),\end{aligned}$$

where we used the integration by parts formula and the decay property of $\varphi \in \mathcal{D}(\mathbb{R})$. We then obtain $(T_{|x|})' = T_{\operatorname{sign}(x)}$.

Hence, the classic derivative and the distributional derivative coincide. Note, however, that while the classical derivative is not defined at the origin $x_0 = 0$, the distributional derivative is a global object, defined in all the domain. One cannot evaluate a distribution in a point, one can only evaluate the duality among a distribution and a test function.

(ii) $T = H(x)$, where $H(x) = \chi_{[0,+\infty)}$ is the Heaviside function.

$$\begin{aligned}\langle T_{H(x)}, \varphi \rangle = -\langle H(x), \varphi' \rangle &= -\int_{\mathbb{R}} H(x)\varphi'(x) \, \mathrm{d}x = -\int_0^{\infty} \varphi'(x) \, \mathrm{d}x = \varphi(0) \\ &= \langle \delta_0, \varphi \rangle \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}).\end{aligned}$$

We then obtain $(T_{H(x)})' = \delta_0$.

(iii) $T = \chi_{[-a,a]}$ for some $a > 0$.

$$\begin{aligned}\langle T_{\chi_{[-a,a]}}, \phi \rangle &= -\langle \chi_{[-a,a]}, \phi' \rangle \\ &= \int_{-a}^{a} -\phi'(x) \, \mathrm{d}x = \phi(-a) - \phi(a) = \langle \delta_{-a}, \phi \rangle - \langle \delta_a, \phi \rangle \qquad \text{for all } \phi \in \mathcal{D}(\mathbb{R}).\end{aligned}$$

So

$$\left(T_{\chi_{[-a,a]}}\right)' = \delta_{-a} - \delta_a,$$

as can be expected by looking at the graph of the function $\chi_{[-a,a]}$: the presence of the two discontinuity points is individuated by the presence of two Delta terms in the distributional derivative centered in the points of the discontinuity.