Exercise Sheet 2

Introduction to Partial Differential Equations (W. S. 2024/25)

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- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am, via moodle.

Exercise 1. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. Let $(u_m)_{m\in\mathbb{N}}$ be a sequence of harmonic functions in Ω, converging uniformly to u. Prove that u is harmonic in Ω.

Solution: Being a uniformly-convergent limit of a sequence of continuous functions, u is continuous on Ω. Thus, in view of the mean-value property, it suffices to show

$$u(x) = \fint_{B_r(x)} u, \qquad \text{for any } B_r(x) \Subset \Omega$$

Since u_m converges uniformly to u, for any $\epsilon > 0$, for sufficiently large m we have

$$\fint_{B_r(x)} |u_m - u| \leq \epsilon \fint_{B_r(x)} 1 = \epsilon$$

Thus, we have $u_m(x) = \fint_{B_r(x)} u_m \to \fint_{B_r(x)} u$ and $u_m(x) \to u(x)$. Due to the uniqueness of the limit, we have $u(x) = \fint_{B_r(x)} u$. Noting that $B_r(x) \Subset \Omega$ is arbitrary, the proof is now complete.

Exercise 2. Prove that the zeros of a harmonic function in $\mathbb{R}^n$ (with $n \geq 2$) cannot be isolated.

Solution: Given an open domain $\Omega \subset \mathbb{R}^n$, let u be harmonic in Ω. Let $x \in \Omega$ be a zero of u. We will show that any open ball centred at x contains a zero other than x. Let $r > 0$ be arbitrary, with $B_r(x) \Subset \Omega$. The mean value property implies that

$$\fint_{\partial B_{r/2}(x)} u = u(x) = 0.$$

Therefore, there must exist $y \in \partial B_{r/2}(x)$ such that $u(y) = 0$, since otherwise the connectedness of $\partial B_{r/2}(x)$ and the continuity of u imply $\fint_{\partial B_{r/2}(x)} u \neq 0$. Hence, $B_r(x)$ contains a zero of u other than x.

Exercise 3. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. Given a harmonic function u, prove that

(i) u^2 is sub-harmonic;

(ii) $|\nabla u|^2$ is sub-harmonic.

Solution:

(i) $\Delta u^2 = \nabla \cdot \left(\nabla u^2\right) = 2\nabla \cdot (u\nabla u) = 2u\Delta u + 2|\nabla u|^2 = 2|\nabla u|^2 \geq 0.$

(ii) Since $u \in C^3(\Omega)$, $\Delta \partial_{x_i} u = \partial_{x_i} \Delta u = 0$ for $i = 1, \ldots, n$; so $\partial_{x_i} u$ is harmonic for $i = 1, \ldots, n$. Then, by Point (i), $(\partial_{x_i} u)^2$ is sub-harmonic. Then the claim in Point (ii) follows from the linearity of the Laplacian operator.

Exercise 4. Let u be harmonic in $\mathbb{R}^n$ (with $n \geq 2$).

(i) Prove that, if $\int_{\mathbb{R}^n} u^2 < \infty$, then $u = 0$.

(ii) Prove that, if $\int_{\mathbb{R}^n} |\nabla u|^2 < \infty$, then u is constant.

Solution: Let $\psi \in C^2(\mathbb{R}^n)$ be a non-negative subharmonic function in $\mathbb{R}^n$. Since ψ is subharmonic, then the mean-value property yields $\psi(x) \leq \fint_{\partial B_r(x)} \psi$ for all $x \in \mathbb{R}^n$ and $r > 0$.

Let $\omega_n = \int_{\partial B_1} 1$, so that

$$\fint_{\partial B_r(x)} \psi = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} \psi$$

If $\int_{\mathbb{R}^n} \psi < \infty$, we have for all $x \in \mathbb{R}^n$

$$\infty > \int_{\mathbb{R}^n} \psi = \int_0^\infty \left(\int_{\partial B_r(x)} \psi \right) \mathrm{d}r \geq \psi(x)\omega_n \int_0^\infty r^{n-1} \, \mathrm{d}r$$

Since $\omega_n > 0, \psi(x)$ must be zero.

By Exercise 3 and harmonic regularity, u^2 and $|\nabla u|^2$ are $C^2(\mathbb{R}^n)$ and sub-harmonic in $\mathbb{R}^n$, and thus choosing $\psi = u^2$ immediately yields the claim in Point (i).

To prove the claim in Point (ii), we take $\psi = |\nabla u|^2$, so that $\nabla u = 0$ in $\mathbb{R}^n$, and (ii) follows by continuity.