# Exercise Sheet 1

# Introduction to Partial Differential Equations (W. S. 2024/25)

EPFL, Mathematics section, Dr. Nicola De Nitti

- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am, via moodle.

**Exercise 1.** Determine whether the following functions are harmonic in their domains of definition:

$$u(x, y) = e^x \cos(y)$$
$$v(x, y) = e^x \sin(y)$$
$$w(x, y) = \ln\left(x^2 + y^2\right)$$

**Solution:** We compute

$$\begin{aligned}
\Delta u &= \frac{\partial^2 \left(e^x \cos(y)\right)}{\partial x^2} + \frac{\partial^2 \left(e^x \cos(y)\right)}{\partial y^2} = e^x \cos(y) - e^x \cos(y) = 0; \\
\Delta v &= \frac{\partial^2 \left(e^x \sin(y)\right)}{\partial x^2} + \frac{\partial^2 \left(e^x \sin(y)\right)}{\partial y^2} = e^x \sin(y) - e^x \sin(y) = 0; \\
\Delta w &= \frac{4}{x^2 + y^2} - \frac{4x^2}{\left(x^2 + y^2\right)^2} - \frac{4y^2}{\left(x^2 + y^2\right)^2} \\
&= \frac{4x^2 + 4y^2}{\left(x^2 + y^2\right)^2} - \frac{4x^2}{\left(x^2 + y^2\right)^2} - \frac{4y^2}{\left(x^2 + y^2\right)^2} = 0.
\end{aligned}$$

**Exercise 2.** Let $\Omega$ be an open bounded domain of $\mathbb{R}^n$ with a smooth boundary.

(i) Let $u : \bar{\Omega} \to \mathbb{R}$ and $\mathbf{v} : \bar{\Omega} \to \mathbb{R}^n$ be $C^1$-functions. Prove the following integration by parts formula:

$$\int_\Omega u \operatorname{div}(\mathbf{v}) = \int_{\partial\Omega} u\mathbf{v} \cdot \nu - \int_\Omega \nabla u \cdot \mathbf{v},$$

where $\nu$ denotes the exterior normal of $\partial\Omega$.

**Hint:** We take the *divergence theorem* for granted: $\int_\Omega \operatorname{div}(u\mathbf{v}) = \int_{\partial\Omega} u\mathbf{v} \cdot \nu$.

(ii) Let $u, w \in C^2(\bar{\Omega})$. Prove that

$$\begin{aligned}
\int_\Omega u \Delta w &= -\int_\Omega \nabla u \cdot \nabla w + \int_{\partial\Omega} u \partial_\nu w \\
&= \int_\Omega w \Delta u + \int_{\partial\Omega} \left(u \partial_\nu w - w \partial_\nu u\right)
\end{aligned}$$

(iii) Let $v \in C^2(\bar{\Omega})$ be harmonic. Prove that

$$\int_{\partial\Omega} \partial_\nu v = 0$$

**Solution:**

(i) We notice that $\nabla \cdot (u\mathbf{v}) = u\nabla \cdot \mathbf{v} + \nabla u \cdot \mathbf{v}$, and use the divergence theorem $\int_\Omega \nabla \cdot (u\mathbf{v}) = \int_{\partial\Omega} u\mathbf{v} \cdot \nu$.

(ii) For the first identity, choose $\mathbf{v} := \nabla w$ in (i). For the second identity, interchange the roles of $u$ and $w$ in the first identity $\int_\Omega u\Delta w = -\int_\Omega \nabla u \cdot \nabla w + \int_{\partial\Omega} u\partial_\nu w$ and subtract the two identities.

(iii) Choosing $u = 1$ and $w = v$ in the first identity of point (ii) yields

$$0 = \int_\Omega \Delta v = \int_{\partial\Omega} \partial_\nu v$$

**Exercise 3.** Let $\Omega \subset \mathbb{R}^n$ be open, bounded, connected, and with a smooth boundary $\partial\Omega$. Let $u \in C^2(\bar{\Omega})$ a harmonic function in $\Omega$.

(i) Prove that, if $\partial_\nu u = 0$ on $\partial\Omega$, then $u$ is constant.

(ii) Prove that, if $u = 0$ on $\partial\Omega$, then $u = 0$.

**Solution:** Applying the first identity in Exercise 2-(ii), with $u \equiv w$, we deduce

$$\int_\Omega |\nabla u|^2 = \int_{\partial\Omega} u\partial_\nu u - \int_\Omega u\Delta u = \int_{\partial\Omega} u\partial_\nu u$$

which is equal to 0 for both cases (i) and (ii).

From this, we deduce that $\nabla u \equiv 0$ a.e. on $\Omega$; by continuity, $\nabla u \equiv 0$ everywhere on $\Omega$. Hence, $u$ is constant in any connected subset of $\Omega$. Moreover, if $u|_{\partial\Omega} = 0$, then we have $u = 0$ from $u \in C(\bar{\Omega})$.

**Exercise 4.**

(i) Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix, and $u : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$ be $C^1$-functions. Let $\tilde{u}(x) := u(Ax)$ and $\tilde{\mathbf{v}}(x) := \mathbf{v}(Ax)$. Prove that

$$\nabla\tilde{u}(x) = A^\top \nabla u(Ax) \quad \text{and} \quad \nabla \cdot \tilde{\mathbf{v}}(x) = [\nabla \cdot (A\mathbf{v})](Ax).$$

(ii) Let $Q \in \mathbb{R}^{n\times n}$ be an orthogonal matrix[1] and $u : \mathbb{R}^n \to \mathbb{R}$ be a harmonic function. Show that $\tilde{u}(x) := u(Qx)$ is harmonic.

[1] Recall that $Q$ is orthogonal if $Q^\top Q = QQ^\top = \mathbb{I}$, where $Q^\top$ denotes the transpose of $Q$ and $\mathbb{I}$ is the identity matrix.

**Solution:**

(i) Let $y := Ax$. Then, we have $y_j = \sum_{k=1}^d A_{jk} x_k$, so that

$$\frac{\partial \tilde{u}}{\partial x_\ell}(x) = \sum_{j=1}^d \frac{\partial \tilde{u}}{\partial y_j}(y) \frac{\partial (Ax)_j}{\partial x_\ell}(x) = \sum_{j=1}^d A_{j\ell} \frac{\partial \tilde{u}}{\partial y_j}(y) \text{ for } \ell = 1, \dots, d,$$

which is the first statement.

Moreover, from

$$\frac{\partial \tilde{v}_\ell}{\partial x_\ell}(x) = \sum_{j=1}^d \frac{\partial v_\ell}{\partial y_j}(y) \frac{\partial (Ax)_j}{\partial x_\ell}(x) = \sum_{j=1}^d A_{j\ell} \frac{\partial v_\ell}{\partial y_j}(y) \quad \text{for} \quad \ell = 1, \dots, d,$$

we have

$$\nabla \cdot \tilde{\mathbf{v}}(x) = \sum_{j=1}^d \sum_{\ell=1}^d A_{j\ell} \frac{\partial v_\ell}{\partial y_j}(y) = \sum_{j=1}^d \frac{\partial}{\partial y_j} \left( \sum_{\ell=1}^d A_{j\ell} v_l(y) \right) = [\nabla \cdot (A\mathbf{v})](Ax)$$

(ii) Let $\mathbf{v}(x) := \left[ Q^\top \nabla u \right](x)$ and $\tilde{\mathbf{v}}(x) := \mathbf{v}(Qx)$. Then, from point (i), we have

$$\begin{aligned} \Delta \tilde{u}(x) &= [\nabla \cdot (\nabla \tilde{u})](x) = \left[ \nabla \cdot \left( Q^\top \nabla u(Q \cdot) \right) \right](x) = (\nabla \cdot \tilde{\mathbf{v}})(x) \\ &= [\nabla \cdot (Q\mathbf{v})](Qx) = \left[ \nabla \cdot \left( Q Q^\top \nabla u \right) \right](Qx) = \Delta u(Qx) = 0. \end{aligned}$$