Exercise Sheet 13

Introduction to Partial Differential Equations (W. S. 2024/25)

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- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

Exercise 1. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Given $\mathbf{u}, \mathbf{v} \in \left[H^1(\Omega)\right]^n$, define the linear strain tensor (or symmetrized gradient)

$$\mathbb{R}^{n\times n} \ni e(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top\right) \qquad \text{i.e.} \qquad (c(\mathbf{u}))_{ij} = \frac{1}{2}\left(\partial_{x_i} u_j + \partial_{x_j} u_i\right),$$

and the bilinear form

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega e(\mathbf{u}) : e(\mathbf{v}) =: \int_\Omega \sum_{i,j=1}^n (e(\mathbf{u}))_{ij} (e(\mathbf{v}))_{ij}.$$

(i) Prove that a is continuous on $\left[H_0^1(\Omega)\right]^n \times \left[H_0^1(\Omega)\right]^n$.

(ii) Show (a special case of) Korn's inequality:[1] there exists $\kappa > 0$ (independent of $\mathbf{u}$) such that

$$\kappa a(\mathbf{u}, \mathbf{u}) \geq \| \|\nabla \mathbf{u}\|_F \|_{L^2(\Omega)}^2 \qquad \text{for all } \mathbf{u} \in \left[H_0^1(\Omega)\right]^{\mathbf{n}}$$

where $\|A\|_F = (A : A)^{1/2} = \left(\sum_{i,j-1}^n A_{ij}^2\right)^{1/2}$ denotes the *Frobenius norm of* $A \in \mathbb{R}^{n\times n}$.[2]

Hint: Approximate $\mathbf{u}$ by smooth functions in order to integrate by parts.

(iii) Let $\mathrm{f} \in \left(\left[H_0^1(\Omega)\right]^n\right)'$. Show that the equation

$$a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \qquad \text{for all } \mathbf{v} \in \left[H_0^1(\Omega)\right]^{\mathbf{n}}$$

is well posed, where $\langle \cdot, \cdot \rangle$ denotes the pairing between $\left[H_0^1(\Omega)\right]^n$ and $\left(\left[H_0^1(\Omega)\right]^n\right)'$.

Solution:

(i) We endow the space $\left[H_0^1(\Omega)\right]^n$ with the $\left[H^1(\Omega)\right]^n$ Sobolev semi-norm

$$\|\mathbf{u}\|_{\left[H_0^1(\Omega)\right]^n} = |\mathbf{u}|_{[H^1(\Omega)]^{\mathbf{n}}} = \| \|\nabla \mathbf{u}\|_F \|_{L^2(\Omega)},$$

[1] Named after Arthur Korn [Kor08, Kor09].
[2] Named after Ferdinand Georg Frobenius.

which can be shown to be a norm by applying (component-wise) the Poincaré inequality. Let $\mathbf{u}, \mathbf{v} \in \left[H_0^1(\Omega)\right]^{\mathbf{n}}$. Since, by the triangular inequality, we have

$$\|e(\mathbf{v})\|_F = \frac{1}{2}\left\|\nabla\mathbf{v} + (\nabla\mathbf{v})^\top\right\|_F \leq \frac{1}{2}\left(\|\nabla\mathbf{v}\|_F + \left\|(\nabla\mathbf{v})^\top\right\|_F\right) = \|\nabla\mathbf{v}\|_F,$$

it suffices to apply the triangular, Cauchy–Schwarz, and Hölder inequalities, to obtain

$$|a(\mathbf{u},\mathbf{v})| \leq \int_\Omega \|\nabla\mathbf{u}\|_F\|\nabla\mathbf{v}\|_F \leq \|\|\nabla\mathbf{u}\,\|_F\|_{L^2(\Omega)}\,\|\|\nabla\mathbf{v}\,\|_F\|_{L^2(\Omega)} = \|\mathbf{u}\|_{\left[H_0^1(\Omega)\right]^{\mathbf{n}}}\|\mathbf{v}\|_{\left[H_0^1(\Omega)\right]^n}$$

(ii) Let $\phi \subset [C_0^\infty(\Omega)]^n \subset \left[H_0^1(\Omega)\right]^n$. We have

$$\begin{aligned} a(\phi,\phi) &= \frac{1}{4}\int_\Omega \left\|\nabla\phi + (\nabla\phi)^\top\right\|_F^2 = \frac{1}{4}\int_\Omega\left(\|\nabla\phi\|_F^2 + \left\|(\nabla\phi)^\top\right\|_F^2 + 2\nabla\phi : (\nabla\phi)^\top\right) \\ &= \frac{1}{2}\int_\Omega \|\nabla\phi\|_F^2 + \frac{1}{2}\int_{\Omega_{i,j-1}}\sum_{x_i}^n \partial_j\partial_{x_j}\phi_i = \frac{1}{2}\|\phi\|^2_{\left[H_0^1(\Omega)\right]^n} + \frac{1}{2}\sum_{i,j-1}^n\int_\Omega \partial_{x_i}\phi_j\partial_{x_j}\phi_{i.}. \end{aligned}$$

Since ϕ is smooth and vanishes at the boundary $\partial\Omega$, we can integrate by parts (twice) the terms in the sum:

$$\int_\Omega \partial_{x_i}\phi_j\partial_{x_j}\phi_i = -\int_\Omega \phi_j\partial_{x_ix_j}\phi_i = \int_\Omega \partial_{x_j}\phi_j\partial_{x_i}\phi_i,$$

so that

$$\sum_{i,j=1}^n\int_\Omega \partial_{x_i}\phi_j\partial_{z_j}\phi_i = \int_\Omega\left(\sum_{i=1}^n \partial_{z_i}\phi_i\right)\left(\sum_{j=1}^n \partial_{x_j}\phi_j\right) = \int_\Omega (\nabla\cdot\phi)^2 \geq 0$$

In summary, $2a(\phi,\phi) \geq \|\phi\|^2_{\left[H_0^1(\Omega)\right]^n}$ for all $\phi \subset [C_0^\infty(\Omega)]^n$. Let $\mathbf{u} \in \left[H_0^1(\Omega)\right]^n$, and consider a sequence $(\phi_m)_{m\geq 1} \subset C_0^\infty(\Omega) \subset \left[H_0^1(\Omega)\right]^n$, converging in $\left[H^1(\Omega)\right]^n$ to $\mathbf{u}$. We remark that the $\left[H^1(\Omega)\right]^n$ norm is equivalent to the $\left[H_0^1(\Omega)\right]^n$ one, so ϕ_m converges to $\mathbf{u}$ in $\left[H_0^1(\Omega)\right]^n$ as well. Since, as shown in Part (i), a is continuous in $\left[H_0^1(\Omega)\right]^n$, we obtain the desired result:

$$2a(\mathbf{u},\mathbf{u}) = \lim_{m\to\infty} 2a\left(\phi_m,\phi_m\right) \geq \lim_{m\to\infty}\|\phi_m\|^2_{\left[H_0^1(\Omega)\right]^{\mathbf{n}}} = \|\mathbf{u}\|^2_{\left[H_0^1(\Omega)\right]^{\mathbf{n}}}$$

The coercivity of a follows, with coercivity constant $1/2$.

(iii) In view of the Lax–Milgram's theorem, the statement is a consequence of Parts (i) and (ii).

Exercise 2. Let Ω be a bounded open subset of $\mathbb{R}^n$, with $n \leq 3$, with smooth boundary. Consider the nonlinear partial differential equation

$$\begin{cases} -\Delta u + u^3 = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^2(\Omega)$.

(i) Derive the weak formulation

$$\begin{aligned}&\text{Find } u \in H_0^1(\Omega) \text{ such that}\\ &a(u,v) := \int_\Omega \nabla u \cdot \nabla v + u^3 v = \int_\Omega f v =: l(v) \quad \text{for all } v \in H_0^1(\Omega),\end{aligned}$$

and show that $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is well-defined, coercive and continuous.

(ii) Consider the subspaces $V^n := \operatorname{span}(w_i)_{i=1}^n \subset H_0^1(\Omega)$, where $\{w_i\}_{i=1}^\infty$ is an Hilbert basis. For every n, consider the map $F^n : V^n \to V^n$ which associates to every $w \in V^n$, the unique element $F^n(w)$ such that

$$(F^{\mathrm{n}}(w), v)_{H_0^1(\Omega)} = a(w,v) - l(v) \quad \text{for all } v \in V^n.$$

Show that F^n is continuous as a map from V^n to V^n.

(iii) Show that the problem

$$\begin{aligned}&\text{Find } u^n \in V^n \text{ such that}\\ &a(u^n, v) = l(v) \quad \text{for all } v \in V^n,\end{aligned}$$

admits a solution u^n which satisfies $\|u^n\|_{H_0^1(\Omega)} \leq \|I\|_{H^{-1}(\Omega)}$.

Hint: Use the following corollary to Browder's fixed point theorem:[3] Let $(V, \|\cdot\|)$ be a finite-dimensional normed vector space and let $f : V \to V'$ be a continuous mapping with the following property: There exists $r > 0$ such that

$$\langle f(v), v \rangle \geq 0 \quad \text{for all } v \in V \text{ such that } \|v\| = r.$$

Then there exists $v_0 \in V$ such that $\|v_0\| \leq r$ and $f(v_0) = 0$.

(iv) Show that the sequence $\{u^n\}_{n\in\mathbb{N}}$ admits a subsequence which converges weakly to u which is solution of the weak formulation above, and further $\|u\|_{H_0^1(\Omega)} \leq \|l\|_{H^{-1}(\Omega)}$.

Solution:

(i) The form $a(\cdot,\cdot)$ is clearly coercive.

We next focus on the continuity, that is we want to show that if $u^n \to u$ and $v^n \to v$ in $H_0^1(\Omega)$ as $n \to \infty$, then $|a(u^n, v^n) - a(u,v)| \to 0$. First, we remark that

$$|a(u^n, v^n) - a(u,v)| \leq |a(u^n, v^n) - a(u^n, v)| + |a(u^n, v) - a(u,v)|.$$

On the one hand, $\left|a\left(u^n, v^k\right) - a(u^n, v)\right| \to 0$ as $n \to \infty$, since $a(u^n, \cdot)$ is linear and bounded, thus continuous with respect to the second argument. On the other hand,

$$|a(u,v) - a(u^n, v)| \leq \int_\Omega |\nabla(u^n - u) \nabla v| + \int_\Omega \left|\left((u^n)^3 - u^3\right) v\right|.$$

[3] Named after Felix Browder [Bro65]

Only the second term is troublesome. However, we have

$$\int_\Omega \left| \left((u^n)^3 - u^3 \right) v \right| \leq \int_\Omega \left| (u^n - u) \left((u^n)^2 + u^2 + u^n u \right) v \right|.$$

We then show the details for a single term, the others being similar:

$$\int_\Omega \left| (u^n - u)\, u^2 v \right| \leq \|u^n - u\|_{L^4(\Omega)} \left\|u^2\right\|_{L^2(\Omega)} \|v\|_{L^4(\Omega)},$$

which tends to zero using the continuous embedding of $H^1_0(\Omega)$ into $L^4(\Omega)$. It will be useful for the last point to remark that $u^n \to u$ in $L^4(\Omega)$ is sufficient to have the continuity of the nonlinear term.

(ii) Fixed a $w \in V^n$, we have

$$\begin{aligned} |l_w(v)| := |a(w,v) - l(v)| &\leq \left(C_1 \|w\|_{H^1_0(\Omega)} + C_2 \|w\|^3_{H^1_0(\Omega)} \right) \|v\|_{H^1_0(\Omega)} + \|f\|_{H^{-1}(\Omega)} \|v\|_{H^1_0(\Omega)} \\ &\leq C(w,f) \|v\|_{H^1_0(\Omega)}, \end{aligned}$$

hence $l_w(\cdot)$ is a linear functional over V^n, thus the existance and uniqueness of $F^n(w)$ follows from Riesz's representation theorem. We show that l_w is continuous with respect to w. Let $\{w_k\}_{k \subset N} \subseteq V_n$ be a sequence such that $w_k \to w$ in V^n for some $w \in V^n$, then, thanks to the continuity of $a(\cdot,\cdot)$ (with respect the first argument), $l_{w_k} \to l_w$ in $(V^n)'$. This means that $\|l_{w_k} - l_w\|_{(V^n)'} \to 0$. Now consider the Riesz' map $J : (V^n)^* \to V^n$, that is, a linear map such that

$$\langle \Lambda_1, \Lambda_2 \rangle_* = (J\Lambda_1, J\Lambda_2)_{H^1_0(\Omega)} \quad \text{for all } \Lambda_1, \Lambda_2 \in (V^n)'$$

The continuity of $F^n : V^n \to F^n$ follows by direct computation

$$\begin{aligned} 0 \leftarrow \langle l_{w_k} - l, L_{w_k} - l \rangle_* = \|l_{w_k} - l\|_{(V^n)'} &= \left(J\left(l_{w_k} - l\right), J\left(l_{w_k} - l\right) \right)_{H^1_0(\Omega)} \\ &= \left(F^n\left(w_k\right) - F^n(w), F^n\left(w_k\right) - F^n(w) \right)_{H^1_0(\Omega)} \\ &= \|F^n\left(w_k\right) - F^n(w)\|^2_{H^1_0(\Omega)} \end{aligned}$$

This shows that, since l_w is continuous with respect to w, we have that F^n is continuous as well.

(iii) It is sufficient to apply the corollary with $f := F^n$. Indeed F^n is continuous from Part (ii), and further

$$(F^n(w), w)_{H^1_0(\Omega)} = a(w,w) - l(w) \geq \|u\|^2_{H^1_0(\Omega)} - \|l\|_{H^{-1}(\Omega)} \|u\|_{H^1_0(\Omega)} \geq 0$$

if $\|u\|_{H_g(\Omega)} = \|f\|_{H^{-1}(\Omega)}$. Consequently, there exists u^n such that $\|u^n\|_{H_0(\Omega)} \leq \|l\|_{H^{-1}(\Omega)}$, and further $F^n\left(u^n\right) = 0$ which implies $a\left(u^n, v\right) = l(v)$, for every $v \in V^n$.

(iv) The sequence $\{u^n\}_{n-1}^\infty$ is bounded in norm $H^1_0(\Omega)$, hence it admits a weakly convergent subsequence, that is $u^{n_k} \rightharpoonup u$, which satisfies $\|u\|_{H^1_0(\Omega)} \leq \liminf_{k\to\infty} \|u^{n_k}\|_{H^1_0(\Omega)} \leq \|f\|_{H^{-1}(\Omega)}$. Furthermore using the compact embedding of $H^1_0(\Omega)$ in $L^4(\Omega)$, we can extract a subsequence of u^n which converges strongly in $L^4(\Omega)$ to u. To conclude, given any element $v \in H_0^1(\Omega)$ take $v^n \in V^n$ s.t. $\|v^n - v\|_{H_0'(\Omega)} \to 0$, and remark

$$0 = \lim_{k\to\infty} a\left(u^{n_k}, v^{n_k}\right) - l\left(v^{n_k}\right) = a(u,v) - l(v)$$

which follows from part (i) and shows that u satisfies the weak formulation.

References

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