Exercise Sheet 11
Introduction to Partial Differential Equations (W. S. 2024/25)
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- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

Exercise 1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and consider the uniformly elliptic operator

$$Lu(x) = -\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x).$$

Given a continuous function $\phi : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, non-decreasing with respect to the second argument (i.e., $\phi(x,p) \leq \phi(x,q)$ for all $x \in \overline{\Omega}$, $p, q \in \mathbb{R}$, $p < q$), we define the semilinear operator

$$[Q(u)](x) = Lu(x) + \phi(x, u(x)).$$

(i) Let $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ be such that we have $Q(u) \leq Q(v)$ in Ω and $u \leq v$ on $\partial\Omega$. Show that $u \leq v$ in Ω.

Hint: Show that $\Omega' = \{x \in \Omega : \, u(x) > v(x)\}$ is empty by applying a maximum principle for L.

(ii) Let $f : \Omega \to \mathbb{R}$. Show that, if the Dirichlet problem

$$\begin{cases} Q(u) = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

admits a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$, then it is unique.

(iii) Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of the Dirichlet problem above. Show an a priori bound of the form

$$\max_{\overline{\Omega}} |u| \leq C \left(\sup_{\Omega} |f| + \max_{\overline{\Omega}} |\phi(\cdot, 0)| \right).$$

Hint: Define $\psi(x) = C_0 - C_1|x|^2$, with C_0 and C_1 chosen in such a way that $\psi \geq 0$ in Ω, $Q(\psi) \geq Q(u)$ in Ω, and $\psi \geq u$ on $\partial\Omega$.

Solution:

(i) Let $w = u - v$ and $\Omega^+ = \{x \in \Omega : w(x) > 0\}$, which is open by the continuity of w. If $\Omega^+ = \emptyset$, the claim follows. Conversely, assume that there exists $y \in \Omega^+$; then

$$0 \geq Q(u)(y) - Q(v)(y) = Lw(y) + \underbrace{\phi(y, u(y)) - \phi(y, v(y))}_{\geq 0, \ \text{since } u(y) > v(y)}.$$

Hence, $Lw \leq 0$ in Ω^+, and we can employ the maximum principle to obtain

$$\max_{\partial\Omega^+} w = \max_{\overline{\Omega^+}} w = \max_{\overline{\Omega}} w,$$

where the last equality follows from $w(x) \leq 0 \leq \max_{\overline{\Omega^+}} w$, for all $x \in \overline{\Omega} \backslash \overline{\Omega^+}$. Observe that $\partial\Omega^+$is a subset of $\partial\Omega \cup \Omega^0$, with $\Omega^0 = \{x \in \Omega : w(x) = 0\}$. But $w \leq 0$ on $\partial\Omega^+$, hence $\max_{\overline{\Omega}} w \leq 0$, in contradiction with the hypothesis $\Omega^+ \neq \emptyset$.

(ii) Let $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ be solutions of the Dirichlet problem. Then $Q(u) = Q(v)$ in Ω, and $u = v$ on $\partial\Omega$. From the previous point we can conclude $u = v$ in $\overline{\Omega}$.

(iii) Let $R = \max_{x \in \overline{\Omega}} |x|$, and define $\psi(x) = C\left(R^2 - |x|^2\right)$, which, in particular, satisfies $\psi \geq 0$ in $\overline{\Omega}$, for every $C \geq 0$. We will fix C in the following. For all $x \in \Omega$,

$$Q(\psi)(x) = 2C \sum_{i=1}^{n} a_{ii}(x) + \phi(x, \psi(x)) \geq 2Cn\lambda + \phi(x, 0) \geq 2Cn\lambda - \max_{\overline{\Omega}} |\phi(\cdot, 0)|,$$

where $\lambda \in \mathbb{R}$ is such that $0 < \lambda|\xi|^2 \leq \sum_{i,j-1}^{n} a_{ij}\xi_i\xi_j$ in Ω, for all $\xi \in \mathbb{R}^n \backslash \{0\}$. Now, we set $C = \left(\sup_{\Omega} |f| + \max_{\overline{\Omega}} |\phi(\cdot, 0)|\right) / (2n\lambda)$, so that

$$Q(\psi) \geq \sup_{\Omega} |f| \geq f = Q(u) \quad \text{ in } \Omega$$

Moreover, since $u = 0$ on $\partial\Omega$, $\psi \geq u$ on $\partial\Omega$. Then, from Point (i), it must be true that $\psi \geq u$ in Ω, and

$$\max_{\overline{\Omega}} u \leq \max_{\overline{\Omega}} \psi = \max_{x\overline{\Omega}} \frac{\sup_{\Omega} |f| + \max_{\overline{\Omega}} |\phi(\cdot, 0)|}{2n\lambda} \left(R^2 - |x|^2\right) \leq \frac{\sup_{\Omega} |f| + \max_{\overline{\Omega}} |\phi(\cdot, 0)|}{2n\lambda} R^2.$$

A similar derivation involving $-\psi$ shows that

$$\min_{\overline{\Omega}} u \geq -\frac{\sup_{\Omega} |f| + \max_{\overline{\Omega}} |\phi(\cdot, 0)|}{2n\lambda} R^2.$$

Exercise 2. Let $\Omega = B_1(0) \subset \mathbb{R}^n$. Consider $\alpha \in C(\partial\Omega)$, with $\alpha > 0$ on $\partial\Omega$. Also, consider a vector field $\beta : \partial\Omega \to \mathbb{R}^n$, such that $\beta(x) \cdot \nu(x) > 0$ for all $x \in \partial\Omega$, with $\nu(x) = x$ being the usual outward normal vector. Let $u \in C^2(\overline{\Omega})$ satisfy $Lu = 0$ in Ω, with

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i},$$

a uniformly elliptic operator, with a_{ij} and b_i uniformly bounded. Moreover, assume that u satisfies the oblique Robin condition

$$\alpha u + \partial_\beta u := \alpha u + \beta \cdot \nabla u = 0 \quad \text{ on } \partial\Omega.$$

Show that $u = 0$ in $\overline{\Omega}$.

Hint 1: Show that $\max_{\overline{\Omega}} u \le 0 \le \min_{\overline{\Omega}} u$.

Hint 2: it may help to consider a non-negative auxiliary function $\phi(x) = e^{-\gamma|x|^2} - e^{-\gamma}$, with $\gamma > 0$, on an annulus $\Omega \backslash \overline{B_\rho}(0), \rho < 1$.

Solution: First, let u attain its maximum (resp. minimum) over $\overline{\Omega}$ at x_0 (resp. y_0). By the strong maximum principle, if $x_0 \in \Omega$ (resp. $y_0 \in \Omega$), then u is constant over $\overline{\Omega}$. But then the boundary condition would imply that $u = -\partial_\beta u/\alpha = 0$ on $\partial\Omega$, and necessarily $u = 0$ in $\overline{\Omega}$. Next, let $x_0 \in \partial\Omega$ and $y_0 \in \partial\Omega$, with $u(y_0) < u(x) < u(x_0)$ for all $x \in \Omega$. The statement will follow if we can show that $u(x_0) = u(y_0) = 0$, or, equivalently, that $u(x_0) \le 0 \le u(y_0)$. We show $u(x_0) \le 0$ (the argument to show $u(y_0) \ge 0$ is analogous). Assume that $u(x_0) > 0$. Then, by the boundary condition,

$$\partial_\beta u(x_0) = -\alpha(x_0) u(x_0) < 0. \tag{1}$$

We will show $\partial_\beta u(x_0) \ge 0$, from which $u(x_0) \le 0$ will follow. We look for a suitable auxiliary function so that $\partial_\beta u(x_0) = \lim_{\substack{h \to 0 \\ h > 0}} \frac{u(x_0 + h\beta(x_0)) - u(x_0)}{h} > 0$. Let $\gamma > 0$ and $\phi(x) = e^{-\gamma|x|^2} - e^{-\gamma}$ for

$$x \in \Omega' := \Omega \backslash \overline{B_{1/2}}(0)$$

We observe that $\phi > 0$ in Ω', $\phi = 0$ on $\partial\Omega$, and

$$\frac{\partial \phi}{\partial x_i}(x) = -2\gamma e^{-\gamma|x|^2} x_i, \quad \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) = 2\gamma\left(2\gamma x_i x_j - \delta_{ij}\right) e^{-\gamma|x|^2} \quad \text{ for } x \in \Omega'.$$

Hence, in particular,

$$\begin{aligned}
(L\phi)(x) &= -\sum_{i,j=1}^n a_{ij}(x) 2\gamma\left(2\gamma x_i x_j - \delta_{ij}\right) e^{-\gamma|x|^2} + \sum_{i=1}^n b_i(x)(-2\gamma) e^{-\gamma|x|^2} x_i \\
&= -2\gamma e^{\gamma|x|^2}\left(2\gamma \sum_{i,j=1}^n a_{ij}(x) x_i x_j + \sum_{i=1}^n \left(b_i(x) x_i - a_{ii}(x)\right)\right) \\
&\le -2\gamma e^{\gamma|x|^2}\left(2\gamma\bar{\lambda}|x|^2 - |b(x)||x| - n\bar{\Lambda}\right),
\end{aligned}$$

where $0 < \bar{\lambda}|\xi|^2 \le \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \le \bar{\Lambda}|\xi|^2$ in Ω, for all $\xi \in \mathbb{R}^n \backslash \{0\}$. We assume that γ is large enough, so that $L\phi < 0$ in $B_1(0) \backslash \overline{B_{1/2}(0)}$ (e.g., $\gamma > 2(\sup_\Omega |b| + n\bar{\Lambda})/\bar{\lambda}$ suffices). Since, for $x \in \partial B_{1/2}(0)$, $u(x) < u(x_0)$ and $\phi(x) > 0$, there exists $\varepsilon > 0$ such that

$$\psi(x) := u(x) - u(x_0) + \varepsilon\phi(x)$$

satisfies $\psi \le 0$ on $\partial B_{1/2}(0)$. Also, we remark that the inequality $\psi \le 0$ holds true on $\partial\Omega$ as well, since $\phi = 0$ there. Hence, ψ satisfies $L\psi \leq 0$ in Ω', and $\psi \leq 0$ on $\partial\Omega'$. But then the weak maximum principle implies that $\psi \leq 0$ in Ω', i.e., in particular, that

$$u(x) - u(x_0) \leq -\varepsilon\phi(x) = -\varepsilon\left(\phi(x) - \phi(x_0)\right) \qquad \text{for } x \in \Omega \cap B_r(x_0),$$

provided $0 < r < \frac{1}{2}$. Now it suffices to take the derivative in the $\beta(x_0)$-direction at x_0 :

$$\partial_{\beta(x_0)} u(x_0) = \lim_{\substack{h\to 0,\\ h<0}} \frac{u(x_0 + h\beta(x_0)) - u(x_0)}{h} \geq \lim_{\substack{h\to 0,\\ h<0}} -\varepsilon \frac{\phi(x_0 + h\beta(x_0)) - \phi(x_0)}{h} = -\varepsilon \partial_{\beta(x_0)}\phi(x_0)$$

(we take $h < 0$ because we want $x_0 + h\beta(x_0) \in \Omega$, and $\beta(x_0)$ is pointing outwards). A direct computation of $\partial_{\beta(x_0)}\phi(x_0)$ leads to

$$\partial_{\beta(x_0)} u(x_0) \geq -\varepsilon\beta(x_0)\cdot\nabla\phi(x_0) = 2\varepsilon\gamma\left(\beta(x_0)\cdot x_0\right)e^{-\gamma|x_0|^2} = 2\varepsilon\gamma e^{-\gamma}\left(\beta(x_0)\cdot\nu(x_0)\right) > 0$$

in contradiction with (1).

Exercise 3. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. Let u be a smooth solution of the uniformly elliptic equation $Lu := -\sum_{i,j-1}^n a_{ij}(x)u_{z_iz_j} = 0$ in Ω. Assume that the coefficients have bounded first derivatives.

(i) Set $v := |\nabla u|^2 + \lambda u^2$ and show that

$$Lv \leq 0 \quad \text{in } \Omega$$

if λ is large enough.

Hint: Use the fact that $\nabla Lu = 0$.

(ii) Deduce

$$\sup_\Omega |\nabla u| \leq C\left(\max_{\partial\Omega}|\nabla u| + \max_{\partial\Omega}|u|\right).$$

Solution:

(i) We exploit the fact that $\nabla Lu = 0$ in Ω (since $Lu = 0$ in Ω) so that

$$\nabla Lu = \sum_{i,j=1}^n \left[\nabla a_{ij}\right] u_{x_ix_j} + \sum_{i,j=1}^n a_{ij}(\nabla u)_{x_ix_j} = 0.$$

Multiplying both sides by $2\nabla u$ in scalar product yields

$$\begin{aligned}\sum_{i,j=1}^{n} 2\nabla vu \cdot [\nabla a_{ij}]\, u_{x_ix_j} &= -\sum_{i,j=1}^{n} a_{ij} 2\nabla u \cdot (\nabla u)_{x_ix_j} \\ &= -\sum_{i,j=1}^{n} a_{ij} \sum_{k=1}^{n} 2u_{x_k} \left(u_{x_k}\right)_{x_ix_j} \\ &= -\sum_{i,j=1}^{n} a_{ij} \sum_{k=1}^{n} \left[\left(u_{x_k}^2\right)_{x_ix_j} - 2u_{x_kx_i}u_{x_kx_j}\right]\end{aligned} \tag{2}$$

Observing that $-\sum_{i,j-1}^{n} a_{ij} \sum_{k-1}^{n} \left(u_{x_k}^2\right) x_ix_j = -\sum_{i,j-1}^{n} a_{ij} \left(|\nabla u|^2\right)_{x_ix_j}$, (2) yields

$$-\sum_{i,j=1}^{n} a_{ij} \left(|\nabla u|^2\right)_{x_ix_j} = 2\sum_{i,j=1}^{n} \nabla u \cdot [\nabla a_{ij}]\, u_{x_ix_j} - 2\sum_{k=1}^{n}\sum_{i,j=1}^{n} a_{ij}u_{x_kx_k}u_{x_kx_j}.$$

We still need to consider $\lambda \left(u^2\right)_{x_ix_j} = 2\lambda u_{x_i}u_{x_j} + 2\lambda u u_{x_ix_j}$, but $Lu = 0$ implies

$$-\lambda \sum_{i,j=1}^{n} a_{ij} \left(u^2\right)_{x_ix_j} = -2\lambda \sum_{i,j=1}^{n} a_{ij}u_{x_i}u_{x_j} + 2(\lambda u)(Lu) = -2\lambda \sum_{i,j=1}^{n} a_{ij}u_{x_i}u_{x_j}.$$

Therefore, we have

$$Lv = 2\sum_{i,j=1}^{n} \nabla u \cdot [\nabla a_{ij}]\, u_{x_ix_j} - 2\sum_{k=1}^{n}\sum_{i,j=1}^{n} a_{ij}u_{x_kx_i}u_{x_kx_j} - 2\lambda \sum_{i,j=1}^{n} a_{ij}u_{x_i}u_{x_j}.$$

We now exploit the boundedness of derivatives the coefficients, $|\nabla a_{ij}| \leq C$, and the uniform ellipticity of L, that is $\sum_{ij} a_{ij}\xi_i\xi_j \geq \theta|\xi|^2$ for all $\xi \in \mathbb{R}^n$. For the first term we have

$$2\nabla u \cdot [\nabla a_{ij}]u_{x_ix_j} \leq 2|\nabla u||[\nabla a_{ij}]||u_{x_ix_j}| \leq |\nabla u|^2 + C^2|u_{x_{ix_j}}|^2,$$

and for the last two terms we use the uniform ellipticity condition to obtain

$$-2\sum_{k=1}^{n}\sum_{i,j=1}^{n} a_{ij}u_{x_kx_k}u_{x_kx_j} \leq -2\theta\sum_{k=1}^{n} \left|\nabla u_{x_k}\right|^2 = -2\theta\sum_{k=1}^{n}\sum_{l=1}^{n} \left|u_{x_kx_l}\right|^2,$$

and

$$-2\lambda \sum_{i,j=1}^{n} a_{ij}u_{x_i}u_{x_j} \leq -2\theta\lambda|\nabla u|^2$$

Hence, we have

$$Lv \leq \left(C^2 - 2\theta\right) \sum_{i,j=1}^{n} \left|u_{x_ix_j}\right|^2 + \left(n^2 - 2\lambda\theta\right)|\nabla u|^2,$$

and thus (using the smoothness of u up to the boundary) for large λ we have $Lv \leq 0$.

(ii) From $Lv \leq 0$, the weak maximum principle implies

$$\sup_{\Omega} |\nabla u|^2 \leq \sup_{\Omega} \left(|\nabla u|^2 + \lambda u^2\right) \leq \max_{\partial\Omega} |\nabla u|^2 + \lambda \max_{\partial\Omega} u^2,$$

which is the desired result.