Exercise Sheet 10

Introduction to Partial Differential Equations (W. S. 2024/25)

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- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

Exercise 1. Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence in a Banach space X. Show

(i) $x_k \to x$ in X implies $x_k \rightharpoonup x$ in X.

(ii) Prove that there exist sequences that converge weakly to the zero function even though all their terms belong to the unit sphere. Conclude that $x_k \rightharpoonup x$ in X does not imply $x_k \to x$ in X.

Hint: Consider the space $X = L^2(0, 2\pi)$ and as a sequence a well-known complete basis.

Solution:

(i) A direct calculation shows that, for all $f \in X'$,

$$|\langle f, x\rangle - \langle f, x_k\rangle| \leq \|f\|_{X'} \|x - x_k\|_X ,$$

thus $x_k \to x$ implies $\langle f, x_k\rangle \to \langle f, x\rangle$ for every f, that is $x_k \to x$.

(ii) Consider the sequence of functions $u_n(x) = \frac{1}{\sqrt{\pi}} \sin(nx)$, for $x \in (0, 2\pi)$ and $n \in \mathrm{N}$. On the one hand,

$$\|u_n\|^2 = \frac{1}{\pi} \int_0^{2\pi} \sin^2(nx)\,\mathrm{d}x = 1,$$

hence the sequence $\{u_n\}_{nCN}$ belongs to the unit sphere. On the other hand, take any $f \in \big(L^2(0, 2\pi)\big)' \equiv L^2(0, 2\pi)$. Bessel's inequality implies that, for $n \to \infty$,

$$(f, u_n)_{L\cdot(0,2\pi)} = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \sin(nx)\,\mathrm{d}x \to 0.$$

Observing that $(0, f)_{L^2(0,2\pi)} = 0$ for every $f \in L^2(0, 2\pi)$, we conclude that $u_n \rightharpoonup 0$.

Exercise 2. Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. Let $(u_m)_{m\in\mathbb{N}}$ be a bounded sequence in $H^1(\Omega)$. Show that there exists a subsequence $(u_{m_k})_{k\in\mathbb{N}}$ and an element $u \in H^1(\Omega)$ such that, as $k \to \infty$,

$$\begin{aligned} u_{m_k} &\to u \quad \text{in } L^2(\Omega), \\ u_{m_k} &\rightharpoonup u \quad \text{in } H^1(\Omega). \end{aligned}$$

Hint: Recall (without proof) the following result: Let X be a Hilbert space. Suppose that the sequence $(u_m)_{m\in\mathbb{N}} \subset X$ is bounded. Then, there exists a subsequence $(u_{m_k})_{k\in\mathbb{N}}$ of $(u_m)_{m\in\mathbb{N}}$ and $u \in X$ such that

$$u_{m_k} \rightharpoonup u \quad \text{in } X.$$

Solution: From Rellich–Kondrachov's compactness theorem, the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, and thus there exists a subsequence $(u_{m_k})_k$ and $u \in L^2(\Omega)$ such that

$$u_{m_k} \to u \quad \text{in } L^2(\Omega) \quad \text{as } k \to \infty.$$

In particular, we also have weak convergence

$$u_{m_k} \rightharpoonup u \quad \text{in } L^2(\Omega) \quad \text{as } k \to \infty$$

Noting that this subsequence $(u_{m_k})_k$ is also bounded in the Hilbert space $H^1(\Omega)$, by the hint, there exists a subsequence $(u_{m_{k_\ell}})_\ell$ and $\tilde{u} \in H^1(\Omega)$ such that

$$u_{m_{k_\ell}} \rightharpoonup \bar{u} \quad \text{in } H^1(\Omega) \quad \text{as } \ell \to \infty.$$

It remains to show $u = \tilde{u}$. We will show that the weak L^2-limit of $(u_{m_k})_\ell$ is $\tilde{u}$. Let $f \in L^2(\Omega)$ be arbitrary. Then, $\langle f, \rangle_{L^2(\Omega)}$ defines a linear continuous functional on $I^1(\Omega)$, and thus, owing to Riesz representation theorem, there exists $\varphi_f \in H^1(\Omega)$ such that

$$\langle f, v\rangle_{L^2(\Omega)} = \langle \varphi_f, v\rangle_{H^1(\Omega)}, \qquad \text{for any } v \in H^1.$$

Hence, $u_{m_{k_\ell}} \rightharpoonup \bar{u}$ in $H^1(\Omega)$ implies that as $\ell \to \infty$ we have

$$\left\langle f, u_{m_{k_k}} \right\rangle_{L^2(\Omega)} = \left\langle \varphi_f, u_{m_{k_k}} \right\rangle'_{H^1(\Omega)} \to \langle \varphi_f, \tilde{u}\rangle_{H^1(\Omega)} = \langle f, \tilde{u}\rangle_{L^2(\Omega)}$$

Therefore, $\lim_{\ell\to\infty}\langle f, u_{m_{k_\ell}}\rangle_{L^2(\Omega)} = \langle f, \bar{u}\rangle_{L^2(\Omega)}$ for any $f \in L^2(\Omega)$. Therefore, by Riesz' representation theorem, the same convergence holds for any continuous linear functional on $L^2(\Omega)$, and thus we have $u_{m_{k_\ell}} \rightharpoonup \bar{u}$ as $\ell \to \infty$. But, since $u_{m_k} \to u$ in $L^2(\Omega)$ as $k \to \infty$, we have $u_{m_{k_\ell}} \to u$ in $L^2(\Omega)$ as $\ell \to \infty$, and thus we conclude $u = \bar{u}$, which completes the proof.

Exercise 3. Let $\Omega = B_1 \subset \mathbb{R}^n$, $n \geq 1$, and define $\Omega_L = \{x \in \Omega : x_1 < 0\}, \Omega_R = \{x \in \Omega : x_1 > 0\}$, and $\Omega_0 = \{x \in \Omega : x_1 = 0\}$. Consider $u_L \in C^1(\bar{\Omega}_L)$ and $u_R \in C^1(\bar{\Omega}_R)$, such that $u_L = u_R$ on Ω_0. Show that $u = u_L \mathbb{1}_{\Omega_L} + u_R \mathbb{1}_{\Omega_n} \in W^{1,p}(\Omega)$ for all $1 \leq p \leq \infty$.

Solution: By the regularity of u_L and u_R, we have that Du_L and Du_R are well defined and $W^{1,p}$ in their respective domains, for all p. Thus, our first step will be to show that Du is well defined, and equal to $Du = Du_L \mathbb{1}_{\Omega_L} + Du_R \mathbb{1}_{\Omega_n}$. To this end, given $\phi \in C_0^\infty(\Omega)$, we compute

$$\begin{aligned}
&\int_\Omega u \partial_{\chi_i} \phi = \int_{\Omega_L} u_L \partial_{x_i} \phi + \int_{\Omega_n} u_R \partial_{x_i} \phi \\
&= \int_{\Omega_0} u_L(x)\phi(x)\nu_i^L(x)\mathrm{d}S(x) - \int_{\Omega_k} \partial_{x_i} u_L \cdot \phi + \int_{\Omega_0} u u_R(x)\phi(x)\nu_i^R(x)\mathrm{d}S(x) - \int_{\Omega_n} \partial_{x_i} u_R \phi \\
&= \int_{\Omega_0} \left(u_L(x)\nu_i^L(x) + u_R(x)\nu_i^R(x)\right)\phi(x)\mathrm{d}S(x) - \int_\Omega \left(D_{x_i} u_L \mathbb{1}_{\Omega_{L-i}} + D_{x_i} u_R \mathbb{1}_{\Omega_H}\right)\phi.
\end{aligned}$$

We remark that the outer normals satisfy $\nu^L + \nu^R = 0$ on Ω_0, so that the first integral is zero. Now it just remains to verify that $u, Du \in L^p(\Omega)$ for all p. But this follows from

$$\|u\|_{L^p(\Omega)} = \|u_L\|_{L^p(\Omega_L)} + \|u_R\|_{LP(\Omega_n)}$$

and

$$\|Du\|_{L^p(\Omega)} = \|Du_L\|_{L^p(\Omega_k)} + \|Du_R\|_{L^p(\Omega_u)}.$$

Exercise 4. Given $W^{k,p}(\Omega)$, with $k \geq 0$ and $1 \leq p < \infty$, we define the set $W_0^{k,p}(\Omega) \subset W^{k,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the $W^{k,p}(\Omega)$-topology, i.e. $u \in W_0^{k,p}(\Omega)$ iff there exists a sequence $(\phi_m)_{m\geq 1} \subset C_0^\infty(\Omega)$ such that

$$\lim_{m\to\infty} \|\phi_m - u\|_{W^{k,p}(\Omega)} = \lim_{m\to\infty} \sum_{|\alpha|\leq k} \|D^\alpha(\phi_m - u)\|_{L^p(\Omega)} = 0.$$

Let us define the *zero-extension (linear) operator* $\zeta : L^P(\Omega) \to L^P(\mathbb{R}^n)$ as

$$\zeta : u \mapsto \zeta u = \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^n \backslash \Omega \end{cases}$$

We have the following property (proof omitted): Given $u \in L^p(\Omega)$, if $\zeta u \in W^{k,p}(\mathbb{R}^n)$, then $u \in W_0^{k,p}(\Omega)$.

Let $\Omega \subset \mathbb{R}^n$ be an open domain and $1 \leq p < \infty$, and consider the zero-extension operator ζ defined above.

(i) Show that $\zeta\left(W_0^{k,p}(\Omega)\right) \subset W^{k,p}(\mathbb{R}^n)$, i.e. that, for all $u \in W_0^{k-p}(\Omega)$, we have $\zeta u \in W^{k,p}(\mathbb{R}^n)$.

<u>Hint:</u> show that $D\zeta u = \zeta(Du)$. a.e. in $\mathbb{R}^n$.

(ii) Let Ω be smooth, such that $\mathbb{R}^n \backslash \bar{\Omega} \neq 0$. For which values of k do we have $\zeta\left(W^{k,p}(\Omega)\right) \subset W^{k,p}(\mathbb{R}^n)$?

Solution:

(i) Following the definition of $W_0^{k,p}(\Omega)$, let $(\phi_m)_{m\geq 1} \subset C_0^\infty(\Omega)$ be a sequence converging to u in $W^{k,p}(\Omega)$ and also a.e. in Ω (such sequence exists as some subsequence of an arbitrary $W^{k,p}(\Omega)$ converging sequence). Also, let $|\alpha| \leq k$ and $\psi \in C_0^\infty(\mathbb{R}^n)$ be arbitrary. Then, by the definition of α-th weak derivative,

$$\int_{\mathbb{R}^-} D^\alpha \zeta(u)\psi = (-1)^{|\alpha|} \int_{\mathbb{R}^-} \zeta(u) D^\alpha \psi = (-1)^{|\alpha|} \int_\Omega u D^\alpha \psi = \lim_{m\to\infty} (-1)^{|\alpha|} \int_\Omega \phi_m D^\alpha \psi,$$

where the last equality follows from $\phi_m \to u$ in $L^p(\Omega)$ as $m \to \infty$. Now the integration by parts gives

$$\lim_{m\to\infty} (-1)^{|a|} \int_\Omega \phi_m D^\alpha \psi = \lim_{m\to\infty} \int_\Omega D^\alpha \phi_m \psi = \int_\Omega D^\alpha u \psi = \int_{\mathbb{R}^n} \zeta\left(D^\alpha u\right)\psi,$$

where the penultimate step follows from the convergence of $D^\alpha \phi_m$ to $D^\alpha u$ in $L^p(\Omega)$. Thus, we have

$$\int_{\mathbb{R}^n} D^\alpha \zeta(u)\psi = \int_{\mathbb{R}^-} \zeta\left(D^\alpha u\right)\psi \quad \text{for any } \psi \in C_0^\infty\left(\mathbb{R}^n\right).$$

Thus, $D^\alpha \zeta(u) = \zeta\left(D^\alpha u\right)$ a.e. in $\mathbb{R}^n$ by the fundamental lemma of the calculus of variations. Finally, noting that $\zeta\left(D^\alpha u\right) \in D^P\left(\mathbb{R}^n\right)$, we see that the statement follows.

(ii) If $k = 0$, $W^{k,p}(\Omega) = L^p(\Omega)$, and it is obvious that the zero extension of a $L^p(\Omega)$ function is in $L^p\left(\mathbb{R}^n\right)$, as there is no additional contribution to the integral. Otherwise, if $k \geq 1$, take a point x of $\partial\Omega$ where the boundary is C^1 with outer normal ν, such that $x + \varepsilon\nu \notin \Omega$ for $\varepsilon > 0$ small enough. We can find an element of $W^{k,p}(\Omega)$ which behaves like a non-zero constant in a Ω-neighborhood of x : for instance, given small enough ε and r, we can take $\mathbb{1}_{B_r(x)} * \eta_\varepsilon$ (with $*$ denoting the $\mathbb{R}^n$-convolution). The zero-extension of such function has a jump discontinuity across $\partial\Omega$ close to x, so that its ν-directional weak derivative is not well defined in L^1 of a neighborhood of x, hence not well defined in $L^1_{\text{loc}}\left(\mathbb{R}^n\right)$. As such, in this case, $\zeta\left(W^{k,p}(\Omega)\right) \not\subset W^{1,p}(\mathbb{R})$ and thus $\zeta\left(W^{k,p}(\Omega)\right)$ cannot be a subset of $W^{k,p}(\mathbb{R})$.

Exercise 5. Let $\Omega = [-1,1]$ and $1 \leq p < \infty$. Show that $W_0^{2,p}(\Omega) \subsetneq \left(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\right)$.

Hint: You may use the following result (without proof): $u \in W_0^{1,p}([-1,1])$ if and only if the zero-extension ζu of u to $\mathbb{R}$ satisfies $\zeta u \in W^{1,p}(\mathbb{R})$ for $1 \leq p < \infty$.

Solution: Obviously $W_0^{2,p}(\Omega) \subset W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, so we just have to show that there exists $u \in W^{2,p}(\Omega) \cap W_0^{1,P}(\Omega)$ such that $u \notin W_0^{2,p}(\Omega)$. We take

$$u(x) = 1 - x^2$$

(but any function $u \in C^2(\bar{\Omega})$ such that $u(-1) = u(1) = 0 \neq |u'(-1)| + |u'(1)|$ works). We have $Du(x) = -2x$ and $D^2u(x) = -2$. Thus, we have $u \in C^2(\bar{\Omega})$, so that $u \in W^{2,p}(\Omega)$ for all $p \geq 1$.

Let us show that $u \in W_0^{1,p}(\Omega)$. We have $\zeta u \in L^p(\mathbb{R})$, and the weak derivative of ζu is well defined: given $\phi \in C_0^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}} D(\zeta u)\phi = -\int_{\mathbb{R}} \zeta u D\phi = -\int_{-1}^{1} uD\phi = \int_{-1}^{1} Du\phi - (u\phi)|_{-1}^{1} = \int_{\mathbb{R}} \zeta(Du)\phi.$$

Moreover, we have $D(\zeta u) \in L^p(\mathbb{R})$ for all $p \geq 1$, so that in view of the hint we have $u \in W_0^{1,P}(\Omega)$. To show that $u \notin W_0^{2,p}(\Omega)$, we observe that, since the first weak derivative of ζu has jump discontinuities at ± 1, it is not weakly differentiable. Thus, $\zeta u \notin W^{2,p}(\mathbb{R})$, and the statement follows.

Exercise 6. In this exercise, we wish to show that $C^\infty(\bar{\Omega})$ is not always dense in $W^{1,p}(\Omega)$: we need to be very careful about the regularity of the domain.

Let $\mathbb{R}^2 \supset \Omega = B_1 \backslash \{(x,0), x \geq 0\}$, and consider $u(x,y) = u(\rho,\theta) = 0$ in polar coordinates ($0 < \rho < 1$ and $0 < 0 < 2\pi$).

(i) Show that $u \in W^{1,1}(\Omega)$, but $u \notin W^{1,1}(B_1)$.

(ii) Show that, for a smooth $\phi \in C^\infty\left(\overline{B_1}\right)$, the natural norms in the spaces $W^{1,1}(\Omega)$ and $W^{1,1}(B_1)$ coincide.

(iii) Conclude that is not possible that there exists $\phi_m \in C^\infty\left(\overline{B_1}\right)$ such that $\phi_m|_\Omega \to u$ in $W^{1,1}(\Omega)$.

Solution:

(i) The function u is bounded in Ω, so it belongs to $L^1(\Omega)$ and $L^1(B_1)$. If its weak gradient exists, it can be calculated directly:

$$\partial_x u(\rho,\theta) = -\frac{1}{\rho}\sin\theta \quad \text{and} \quad \partial_y u(\rho,\theta) = \frac{1}{\rho}\cos\theta.$$

We can check that we know that $|\nabla u(\rho,\theta)| = \left(\partial_x u(\rho,\theta)^2 + \partial_y u(\rho,\theta)^2\right)^{1/2} = \frac{1}{\rho} = \left(x^2+y^2\right)^{-1/2}$ belongs to $L^1(\Omega)$. So it remains to check whether Du is actually the weak gradient of u in $W^{1,1}(\Omega)$ and $W^{1,1}(B_1)$. We start from the latter: given $\phi \in C_0^\infty(B_1)$, the definition of the x-partial weak derivative reads

$$\begin{aligned}
-\int_{B_1} D_x u\phi = \int_{B_1} uD_x\phi &= \int_0^1\int_0^{2\pi} \rho\theta\left(\cos\theta\partial_\rho\phi(\rho,\theta) - \frac{1}{\rho}\sin\theta\partial_\theta\phi(\rho,\theta)\right)\mathrm{d}\theta\ \mathrm{d}\rho \\
&= \int_0^1\int_0^{2\pi} \rho\theta\cos\theta\partial_\rho\phi(\rho,\theta)\mathrm{d}\theta\ \mathrm{d}\rho - \int_0^1\int_0^{2\pi} \theta\sin\theta\partial_\theta\phi(\rho,\theta)\mathrm{d}\theta\ \mathrm{d}\rho \\
&= \int_0^{2\pi} \theta\cos\theta\left(-\int_0^1 \phi(\rho,\theta)\mathrm{d}\rho\right)\mathrm{d}\theta - \int_0^1\left(-\int_0^{2\pi}\partial_\theta(\theta\sin\theta)\phi(\rho,\theta)\mathrm{d}\theta\right)\mathrm{d}\rho \\
&= \int_0^1\int_0^{2\pi} \sin\theta\phi(\rho,\theta)\mathrm{d}\theta\ \mathrm{d}\rho = -\int_{D_1} \partial_x u\phi
\end{aligned}$$

where all the boundary terms due to integration by parts $\left(\rho\phi(\rho,\theta)|_{\rho=0}^{\rho=1} \text{ and } \theta\sin\ \theta\phi(\rho,\theta)|_{\theta=0}^{\theta=2\pi}\right)$ are zero. Now we consider the y-derivative: if $D_y u \in L_{\text{loc}}(B_1)$, then

$$\begin{aligned}
-\int_{B_1} D_y u\phi = \int_{B_1} uD_y\phi &= \int_0^1\int_0^{2\pi}\rho\theta\left(\sin\theta\partial_\rho\phi(\rho,\theta)+\frac{1}{\rho}\cos\theta\partial_\theta\phi(\rho,\theta)\right)\mathrm{d}\theta\ \mathrm{d}\rho\\
&= \int_0^1\int_0^{2\pi}\rho\theta\sin\theta\partial_\rho\phi(\rho,\theta)\mathrm{d}\theta\ \mathrm{d}\rho + \int_0^1\int_0^{2\pi}\theta\cos\theta\partial_0\phi(\rho,\theta)\mathrm{d}\theta\ \mathrm{d}\rho\\
&= \int_0^{2\pi}\theta\sin\theta\left(-\int_0^1\phi(\rho,\theta)\mathrm{d}\rho\right)\mathrm{d}\theta\\
&\quad+\int_0^1\left(2\pi\phi(\rho,2\pi)-\int_0^{2\pi}\partial_\theta(\theta\cos\theta)\phi(\rho,\theta)\mathrm{d}\theta\right)\mathrm{d}\rho\\
&= 2\pi\int_0^1\phi(\rho,2\pi)\mathrm{d}\rho-\int_0^1\int_0^{2\pi}\cos\theta\phi(\rho,\theta)\mathrm{d}\theta\ \mathrm{d}\rho\\
&= 2\pi\int_0^1\phi(\rho,2\pi)\mathrm{d}\rho-\int_{B_1}\partial_y u\phi
\end{aligned}$$

which, in general, is different from $-\int_{B_1}\partial_y u\phi$ due to the presence of the boundary term on $B_1\backslash\Omega$. Hence, we can conclude that there cannot exist $D_y u\in L_{\text{loc}}(B_1)$ satisfying the definition of weak derivative. Consequently, $u\notin W^{1,1}(B_1)$. On the other hand, the same calculation with $\phi\in C_0^\infty(\Omega)$ shows that $Du=\nabla u\in L^1(\Omega)^2$, since $\phi(B_1\backslash\Omega)=0$.

(ii) Since for $\phi\in C^\infty\left(\overline{B_1}\right)$ its classical and its weak derivative coincide and since the Lebesgue measure of $B_1\backslash\Omega$ is zero, the norm of ϕ in the space $W^{1,1}(B_1)$ and the norm of its restriction $\phi|_\Omega$ in the space $W^{1,1}(\Omega)$ coincide as well.

(iii) Suppose by contradiction that there exists $\{\phi_m\}_{m\in\mathbb{N}}\subseteq C^\infty\left(\overline{B_1}\right)$ such that $\phi_m|_\Omega\to u(x,y)=\theta$ in $W^{1,1}(\Omega)$. Then the sequence $\{\phi_m|_\Omega\}_{m\in\mathbb{N}}$ is Canchy in $W^{1,1}(B_1)$, indeed,

$$\|\phi_m|_\Omega-\phi_n|_\Omega\|_{W^{1,1}(\Omega)}=\|\phi_m-\phi_n\|_{W^{1,1}(B_1)}\to 0 \text{ for } m,n\to\infty$$

By completeness of the space $W^{1,1}(B_1)$, there exists $u^\star\in W^{1,1}(B_1)$ such that $\phi_m\to u^\star$ in $W^{1,1}(B_1)$. But this is a contradiction since, by uniqueness, of the limit, it would imply that there exists an extension of $u\in W^{1,1}(\Omega)$ to $u^\star\in W^{1,1}(B_1)$ and this can not happen. In fact, by point a), we proved that the distributional derivative of u with respect to the coordinate y is not a function. If this extension existed, then its distributional derivative would not coincide with its weak derivative (that is an object in $L^p(B_1)$) and this is not possible.