Exercise Sheet 8

Introduction to Partial Differential Equations (W. S. 2024/25)

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- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

Exercise 1. Let $\Omega = B_1(0) \subset \mathbb{R}^n$, $n \geq 1$.

(i) For which combinations $(\alpha, p) \in (\mathbb{R}\backslash\{0\}) \times [1, \infty]$ does the function $|x|^\alpha$ belong to $L^p(\Omega)$ and/or to $W^{1,p}(\Omega)$?

(ii) For which values $1 \leq p \leq \infty$ does $\log|x|$ belong to $L^p(\Omega)$ and/or to $W^{1,P}(\Omega)$?

(iii) Comment on the local (Ω is bounded) L^p and $W^{1,p}$ regularity of the fundamental solution of the Laplacian Φ in $\mathbb{R}^n, n \geq 2$.

Exercise 2. Let $\Omega \subset \mathbb{R}^n$ be a domain, and $u \in W^{1,p}(\Omega)$ for some $1 \leq p < \infty$. Consider an arbitrary $f \in C^1(\mathbb{R})$, with f' bounded.

(i) Let Ω be bounded. Show that $f \circ u := f(u(\cdot)) \in W^{1,p}(\Omega)$ and that $D(f \circ u) = f'(u)Du$.

(ii) Assuming $f(0) = 0$, extend the chain rule above to the case where Ω is unbounded.

(iii) Show that $u^+ = \max\{u, 0\}, u^- = \min\{u, 0\}$, and $|u|$ belong to $W^{1,p}(\Omega)$ as well.

Hint: use appropriately mollified versions of the functions $\max\{\cdot, 0\}$, etc.

Exercise 3. The application $\|\cdot\|_{k,p} : W^{k,p}(\Omega) \to \mathbb{R}_+$ (defined in the Lecture Notes) is a norm for any $1 \leq p \leq \infty$.

Exercise 4. The normed vector space $\left(W^{k,p}(\Omega), \|\cdot\|_{k,p}\right)$ is a Banach space for every $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. In particular, the space $H^k(\Omega) = W^{k,2}(\Omega)$ is a Hilbert space, for every $k \in \mathbb{N}$, with inner product

$$(f, g)_{H^k} := \sum_{|\alpha| \leq k} \int_\Omega D^\alpha f \cdot D^\alpha g \, \mathrm{d}x.$$

Exercise 5. Let $f \in W^{k,p}(\Omega)$, with $1 \leq p < \infty$, and $f_\epsilon := \eta_\epsilon * f : \Omega \to \mathbb{R}$. Then $f_\epsilon \xrightarrow{\epsilon \to 0} f$ in $L^p(\Omega)$ and $f_\epsilon \xrightarrow{\epsilon \to 0} f$ in $W^{k,p}(K)$, for any $K \subset\subset \Omega$.