Exercise Sheet 7
Introduction to Partial Differential Equations (W. S. 2024/25)
EPFL, Mathematics section, Dr. Nicola De Nitti

- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

Exercise 1. Let us start by recalling the classic Ascoli–Arzelà theorem.[1]

> Let X be a compact metric space. If a sequence $\{f_n\}_{n=1}^{\infty}$ in the space[2] $C(X)$ is bounded[3] and equi-continuous[4], then it has a uniformly convergent subsequence. Moreover, if every subsequence of $\{f_n\}_{n=1}^{\infty}$ itself has a uniformly convergent subsequence, then $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded and equi-continuous.

Keeping Ascoli–Arzelà's theorem in mind, prove the following result.

> *Ascoli–Arzelà-type theorem for harmonic functions.* If $\{u_m\}_{m=1}^{\infty}$ is a sequence of harmonic functions on Ω that is uniformly bounded on each compact subset of Ω, then some subsequence of $\{u_m\}_{m=1}^{\infty}$ converges uniformly on each compact subset of Ω.

Exercise 2. Consider the Neumann problem for the Laplace equation on the half space $\Omega := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$:

$$\begin{cases} -\Delta u = 0, & x \in \Omega, \\ \partial_\nu u = h, & x \in \partial\Omega, \end{cases}$$

with $h \in C_0^2(\partial\Omega)$. Let $N(x,y) := \Phi(x-y) + \Phi(x - y^\star)$, where $y^\star := (y_1, \ldots, y_{n-1}, -y_n)$.

[1] Proven first by Giulio Ascoli [Asc84] (who established the sufficient condition for compactness) and then by Cesare Arzelà [Arz95] (who established also the necessary condition). The generalization to real-valued continuous functions with domain a compact metric space is due to Maurice Fréchet [Fré06].

[2] Our setting is a compact metric space X which you can, if you wish, take to be a compact subset of $\mathbb{R}^n$. Let $C(X)$ denote the space of all continuous functions on X with values in $\mathbb{R}$. In $C(X)$ we always regard the distance between functions f and g in $C(X)$ to be

$$\mathrm{dist}(f,g) := \max\{|f(x) - g(x)| : x \in X\}.$$

It is easy to check that dist, so defined, is a metric (the *max-metric*) on $C(X)$, in which a sequence is convergent iff it converges uniformly on X. Similarly, a sequence in $C(X)$ is Cauchy iff it is Cauchy uniformly on X. Thus the max-metric, which from now on we always assume to be part of the definition of $C(X)$, makes that space complete

[3] A family $\mathcal{F} \subset C(X)$ being *bounded* means that there exists a positive constant $M < \infty$ such that $|f(x)| \le M$ for each $x \in X$ and each $f \in \mathcal{F}$

[4] The family $\mathcal{F} \subset C(X)$ being *equi-continuous* means that for every $\varepsilon > 0$ there exists $\delta > 0$ (which depends only on ε) such that, for $x, y \in X$,

$$d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \qquad \text{for all } f \in \mathcal{F}$$

where d is the metric on X.

(i) Prove that $u(y) = \int_{\partial\Omega} N(x,y)h(x)\,\mathrm{d}x$ is well-defined for all $y \in \Omega$, and satisfies $-\Delta u = 0$ in Ω and $\partial_\nu u = h$ on $\partial\Omega$.

(ii) Prove that for $n \geq 3$ we have $\lim_{|y|\to\infty} u(y) = 0$.

Exercise 3. Let $n \geq 2$ and $f \in C_0^2(\mathbb{R}^n)$. Given Φ the fundamental solution of the Laplace equation in $\mathbb{R}^n$, consider

$$w(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,\mathrm{d}y.$$

(i) Use the change of variable $y = x + z$ to rewrite this quantity as $w(x) = \int_{\mathbb{R}^n} \Phi(z)f(x+z)\,\mathrm{d}z$. Show that $w \in C^2(\mathbb{R}^n)$ and $\partial^2_{x_i x_j} w(x) = \int_{\mathbb{R}^n} \Phi(z)\partial_{x_i x_j} f(x+z)\,\mathrm{d}z$.

(ii) Prove that $-\Delta w = f$ in $\mathbb{R}^n$.

(iii) Show that $\lim_{|x|\to\infty} w(x) = 0$ if $n \geq 3$.

Exercise 4. Let Ω be a bounded domain and $f \in C_c^2(\Omega)$. Then, the Newtonian potential

$$w(y) = \int_\Omega \Phi(x-y)f(x)\,\mathrm{d}x, \qquad y \in \mathbb{R}^n,$$

of f satisfies $w \in C^2(\mathbb{R}^n)$ and $-\Delta w = f$ in Ω.

References

[Arz95] C. Arzelà. Sulle funzioni di linee. *Mem. Accad. Sci. Ist. Bologna Cl. Sci. Fis. Mat.*, 5(5):225–244, 1895.

[Asc84] G. Ascoli. Le curve limite di una varietà data di curve. *Rom. Acc. L. Mem.*, 18(3):521–586, 1884.

[Fré06] M. Fréchet. Sur quelques points du calcul fonctionnel. *Rend. Circ. Mat. Palermo*, 22:1–74, 1906.