Exercise Sheet 6

Introduction to Partial Differential Equations (W. S. 2024/25)

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- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

Exercise 1. Let $T \in \mathcal{D}'(\mathbb{R})$. Prove that $T' = 0$ if and only if T is constant.

Hint: If we assume that T is constant, then we note that $T' = 0$ because φ has compact support. Conversely, suppose that $T' = 0$. Then, for all $\varphi \in C_c^\infty(\mathbb{R})$, $\langle T', \varphi \rangle = -\langle T, \varphi' \rangle = 0$, i.e., T vanishes on all functions of the form φ', where $\varphi \in C_c^\infty(\mathbb{R})$. To prove the result, it is helpful to characterize such functions. In particular, one should prove that

$$\big(\psi = \varphi', \text{ with } \varphi \in C_c^\infty(\mathbb{R})\big) \iff \left(\psi \in C_c^\infty(\mathbb{R}) \text{ and } \int_{\mathbb{R}} \psi(x)\,\mathrm{d}x = 0\right).$$

Exercise 2. Let $\Gamma : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be defined by $\Gamma(x) = \frac{1}{8\pi}|x|^2 \log |x|$.

(i) Show that $\Delta^2\Gamma := \Delta(\Delta\Gamma)$ is equal to 0 in $\mathbb{R}^2 \setminus \{0\}$. Moreover, observe that $\int_{\partial B_r(0)} \partial\nu \Delta\Gamma(x)\,\mathrm{d}S(x) = 1$ for all $r > 0$.

(Extra point) Show that Γ is the fundamental solution of the bilaplacian operator Δ^2.

(ii) Let $u \in C^4(\bar{\Omega})$, with $\Omega \subset \mathbb{R}^2$ an open bounded domain with smooth boundary. Prove that, for all $x \in \Omega$, the following Green-type representation formula holds:

$$u(x) = \int_\Omega \Gamma(x - \cdot)\Delta^2 u - \int_{\partial\Omega} \left(\Gamma(x-\cdot)\partial_\nu \Delta u - \Delta u \partial_\nu \Gamma(x - \cdot) + \Delta\Gamma(x-\cdot)\partial_\nu u - u\partial_\nu \Delta\Gamma(x-\cdot)\right).$$

Exercise 3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary and G be the Green function for the domain Ω.

(i) Fix $x, y \in \Omega$. Write

$$v(z) := G(z, x), \quad w(z) := G(z, y), \quad \text{for } z \in \Omega.$$

For $0 < \varepsilon < |x - y|/2$, prove

$$\int_{\partial B_\varepsilon(x)} (v\partial_\nu w - w\partial_\nu v) = \int_{\partial B_\varepsilon(y)} (w\partial_\nu v - v\partial_\nu w)$$

where ν denotes the inward unit normal vector on $\partial B_\varepsilon(x) \cup \partial B_\varepsilon(y)$.

(ii) Prove

$$G(y, x) = G(x, y) \qquad \text{for all } x, y \in \Omega.$$

Exercise 4. Consider the half-plane $\mathbb{R}^2 \supset \Omega = \mathbb{R} \times |0, \infty|$. Given $y = (y_1, y_2) \in \Omega$, we define by reflection $y^\star = (y_1, -y_2)$.

(i) Given the fundamental solution of the Laplace operator $\Phi(x-y) = -(2\pi)^{-1} \log|x-y|$, show that $G(x,y) = \Phi(x-y) - \Phi(x - y^*)$ is the Green function for Ω.

Hint: You need to show that, for any $y \in \Omega$, $h(x,y) := G(x,y) - \Phi(x-y)$ is harmonic in Ω and that $G(\cdot, y) = 0$ on $\partial\Omega$.

(ii) Derive formally the expression of the corresponding Poisson integral

$$u(y) = -\int_{\partial\Omega} \partial_\nu G(x,y) g(x) \, \mathrm{d}S(x), \tag{PI-hp}$$

where $\partial_\nu G(x,y)$ is the normal derivative of $G(\cdot, y)$ at $x \in \partial\Omega$.

(iii) Prove that, if g is continuous and compactly supported, (PI-hp) actually represents a bounded solution of the Dirichlet problem over Ω, that is, $-\Delta u = 0$ in Ω and $\lim_{\substack{y \to y_0,\\ y \in \Omega}} u(y) = g(y_0)$ for each point $y_0 \in \partial\Omega$.

(iv) Prove that (PI-hp) satisfies the following maximum principle:

$$\sup_{\bar{\Omega}} u = \sup_{\partial\Omega} u.$$

(v) Show that, if g is continuous and compactly supported, $u(y) \to 0$ as $|y| \to \infty$, where u is as in (PI-hp). Moreover, show that u is the unique solution to the problem

$$\begin{cases} -\Delta u(x) = 0, & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega, \end{cases}$$

such that $\lim_{|y| \to \infty} u(y) = 0$.