# Exercise Sheet 5
# Introduction to Partial Differential Equations (W. S. 2024/25)
## EPFL, Mathematics section, Dr. Nicola De Nitti

- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

**Exercise 1.** Prove the following result, in which the fundamental solution $\Phi$ of the Laplace equation is used to derive a representation formula for the point value of a $C^2(\Omega)$ function in terms of its Laplacian and boundary values.

> *Green's representation formula*: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^1$ boundary. For any $u \in C^2(\bar{\Omega})$ and $y \in \Omega$, we have
>
> $$u(y) = \int_{\Omega} \Phi(x-y)(-\Delta u(x))\,\mathrm{d}x - \int_{\partial\Omega} \partial_\nu \Phi(x-y) u(x)\,\mathrm{d}S(x) + \int_{\partial\Omega} \Phi(x-y)\partial_\nu u(x)\,\mathrm{d}S(x). \tag{GRF}$$

Context and hints for the proof: The starting point of the proof is Green's integration by parts identity,

$$\int_{\Omega} (v\Delta u - u\Delta v)\,\mathrm{d}x = \int_{\partial\Omega} (v\partial_\nu u - u\partial_\nu v)\,\mathrm{d}S. \tag{GI}$$

which holds for every $u, v \in C^2(\bar{\Omega})$ and a bounded domain with $C^1$ boundary. Now fix $y \in \Omega$. To prove the (GRF), the idea is to take $v = \Phi(\cdot - y)$ in (GI). However, $x \mapsto \Phi(x-y)$ is not $C^2(\bar{\Omega})$ and, in particular, a singularity occurs at $x = y$. So we need to be more careful: to circumvent this difficulty, we write the identity on the domain $\Omega_\epsilon = \Omega \setminus B_\epsilon(y)$, with $\epsilon$ small enough so that $B_\epsilon(y) \subset \Omega$, and let $\epsilon \to 0$.

**Exercise 2.** Let $u \in C^2(\bar{\Omega})$ be a solution (provided it exists) of the Neumann boundary value problem

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ \partial_\nu u = h, & x \in \partial\Omega, \end{cases} \tag{N-BVP}$$

for a smooth domain $\Omega \subset \mathbb{R}^n$ and smooth data $f, h$ which satisfy the compatibility condition

$$\int_{\Omega} f = -\int_{\partial\Omega} h$$

(i) Fix $y \in \Omega$. Show that the Neumann problem

$$\begin{cases} -\Delta\tilde{\psi}(\cdot, y) = 0, & x \in \Omega, \\ \partial_\nu \tilde{\psi}(\cdot, y) = \partial_\nu \Phi(\cdot - y), & x \in \partial\Omega, \end{cases}$$

has no solution $\tilde{\psi} \in C^2(\bar{\Omega})$.

(ii) Fix $y \in \Omega$. Suppose that $\psi(\cdot, y) \in C^2(\bar{\Omega})$ satisfies

$$\begin{cases} -\Delta\psi(\cdot, y) = 0, & x \in \Omega, \\ \partial_\nu \psi(\cdot, y) = \partial_\nu \Phi(\cdot - y) + \frac{1}{|\partial\Omega|}, & x \in \partial\Omega. \end{cases}$$

Show that we have the following representation formula for the solution $u$ of (N-BVP):

$$u(y) - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} u(x) \, \mathrm{d}S(x) = \int_\Omega f(x) N(x, y) \, \mathrm{d}x + \int_{\partial\Omega} h(x) N(x, y) \, \mathrm{d}S(x),$$

where $N(x, y) := \Phi(x - y) - \psi(x, y)$.

**Exercise 3.** The aim of this exercise is to define basic operations over distributions in $\mathcal{D}'(\mathbb{R})$.

First of all, let us introduce some notation. Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function, we then define the following:

(a) *Translation* of $f$ by $a \in \mathbb{R}$:

$$\tau_a f(x) = f(x + a) \quad \text{for all } x \in \mathbb{R}.$$

(b) *Dilation* of $f$ by $a > 0$:

$$D_a f(x) = f(ax) \quad \text{for all } x \in \mathbb{R}.$$

(c) *Reflection* of $f$:

$$\tilde{f}(x) = f(-x) \quad \text{for all } x \in \mathbb{R}.$$

(d) *Multiplication* by a function $g \in C^\infty(\mathbb{R})$:

$$(fg)(x) = f(x)g(x).$$

We now extend these definitions to elements of the space $\mathcal{D}'(\mathbb{R})$.

Given $T \in \mathcal{D}'(\mathbb{R})$, we can define the following distributions.

(a) The *translation* by $a \in \mathbb{R}$ of $T$ is the distribution $\tau_a T \in \mathcal{D}'(\mathbb{R})$ such that:

$$\langle \tau_a T, \phi \rangle = \langle T, \tau_{-a}\phi \rangle \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}).$$

(b) The *dilation* by $a > 0$ of $T$ is the distribution $D_a T \in \mathcal{D}'(\mathbb{R})$ such that:

$$\langle D_a T, \phi \rangle = \left\langle T, \frac{1}{a} D_{1/a}\phi \right\rangle \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}).$$

(c) The *reflection* of $T$ is the distribution $\check{T} \in \mathcal{D}'(\mathbb{R})$ such that:

$$\langle \check{T}, \phi \rangle = \langle T, \check{\phi} \rangle \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}).$$

(d) The *multiplication* of $T$ by $g \in C^{\infty}(\mathbb{R})$ is a distribution $gT \in \mathcal{D}'(\mathbb{R})$ such that:

$$\langle gT, \phi \rangle = \langle T, g\phi \rangle \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}).$$

Given $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, we can associate to $f$ the distribution $T_f \in \mathcal{D}'(\mathbb{R})$ such that:

$$\langle T_f, \phi \rangle = \int_{\mathbb{R}} f(x)\phi(x)\,\mathrm{d}x \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}).$$

First, check that, for $T_f$ built in such a way, the previous definitions are consistent.
Next, prove the following derivation rules for distributions.

(a) Given $T \in \mathcal{D}'(\mathbb{R})$ and $a \in \ R$, $(\tau_a T)' = \tau_a(T')$.

(b) Given $T \in \mathcal{D}'(\mathbb{R})$ and $a > 0$, $(D_a T)' = a\, D_a(T')$.

(c) Given $T \in \mathcal{D}'(\mathbb{R})$, $(\tilde{T})' = -\tilde{T}'$.

(d) Given $T \in \mathcal{D}'(\mathbb{R})$ and $g \in C^{\infty}(\mathbb{R})$, $(gT)' = g'T + gT'$.

We say that a distribution $T \in \mathcal{D}(\mathbb{R})$ is *even* if $\check{T} = T$ in $\mathcal{D}'(\mathbb{R})$ and we say that a distribution $T \in \mathcal{D}(\mathbb{R})$ is *odd* if $\check{T} = -T$ in $\mathcal{D}'(\mathbb{R})$.
Prove that

(a) $\delta_0$ is an even distribution.

(b) $\delta_0'$ is an odd distribution.

Finally, as further exercise, double-check the computations in the examples and remarks of Section 3 "An interlude about distributions" of the Lecture Notes.