Exercise Sheet 13

Introduction to Partial Differential Equations (W. S. 2024/25)

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- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

Exercise 1. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Given $\mathbf{u}, \mathbf{v} \in \left[H^1(\Omega)\right]^n$, define the linear strain tensor (or symmetrized gradient)

$$\mathbb{R}^{n\times n} \ni e(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top\right) \quad \text{i.e.} \quad (c(\mathbf{u}))_{ij} = \frac{1}{2}\left(\partial_{x_i} u_j + \partial_{x_j} u_i\right),$$

and the bilinear form

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega e(\mathbf{u}) : e(\mathbf{v}) =: \int_\Omega \sum_{i,j=1}^n (e(\mathbf{u}))_{ij} (e(\mathbf{v}))_{ij}.$$

(i) Prove that a is continuous on $\left[H_0^1(\Omega)\right]^n \times \left[H_0^1(\Omega)\right]^n$.

(ii) Show (a special case of) Korn's inequality:[1] there exists $\kappa > 0$ (independent of $\mathbf{u}$) such that

$$\kappa a(\mathbf{u}, \mathbf{u}) \geq \| \|\nabla \mathbf{u}\|_F \|^2_{L^2(\Omega)} \quad \text{for all } \mathbf{u} \in \left[H_0^1(\Omega)\right]^{\mathbf{n}}$$

where $\|A\|_F = (A : A)^{1/2} = \left(\sum_{i,j-1}^n A_{ij}^2\right)^{1/2}$ denotes the *Frobenius norm of* $A \in \mathbb{R}^{n\times n}$.[2]

Hint: Approximate $\mathbf{u}$ by smooth functions in order to integrate by parts.

(iii) Let $\mathrm{f} \in \left(\left[H_0^1(\Omega)\right]^n\right)'$. Show that the equation

$$a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in \left[H_0^1(\Omega)\right]^{\mathbf{n}}$$

is well posed, where $\langle \cdot, \cdot \rangle$ denotes the pairing between $\left[H_0^1(\Omega)\right]^n$ and $\left(\left[H_0^1(\Omega)\right]^n\right)'$.

Exercise 2. Let Ω be a bounded open subset of $\mathbb{R}^n$, with $n \leq 3$, with smooth boundary. Consider the nonlinear partial differential equation

$$\begin{cases} -\Delta u + u^3 = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

[1] Named after Arthur Korn [Kor08, Kor09].
[2] Named after Ferdinand Georg Frobenius.

where $f \in L^2(\Omega)$.

(i) Derive the weak formulation

$$\begin{aligned}&\text{Find } u \in H_0^1(\Omega) \text{ such that}\\ &a(u,v) := \int_\Omega \nabla u \cdot \nabla v + u^3 v = \int_\Omega f v =: l(v) \quad \text{for all } v \in H_0^1(\Omega),\end{aligned}$$

and show that $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is well-defined, coercive and continuous.

(ii) Consider the subspaces $V^n := \operatorname{span}(w_i)_{i=1}^n \subset H_0^1(\Omega)$, where $\{w_i\}_{i=1}^\infty$ is an Hilbert basis. For every n, consider the map $F^n : V^n \to V^n$ which associates to every $w \in V^n$, the unique element $F^n(w)$ such that

$$(F^{\mathrm{n}}(w), v)_{H_0^1(\Omega)} = a(w,v) - l(v) \quad \text{for all } v \in V^n.$$

Show that F^n is continuous as a map from V^n to V^n.

(iii) Show that the problem

$$\begin{aligned}&\text{Find } u^n \in V^n \text{ such that}\\ &a(u^n, v) = l(v) \quad \text{for all } v \in V^n,\end{aligned}$$

admits a solution u^n which satisfies $\|u^n\|_{H_0^1(\Omega)} \le \|I\|_{H^{-1}(\Omega)}$.

Hint: Use the following corollary to Browder's fixed point theorem:[3] Let $(V, \|\cdot\|)$ be a finite-dimensional normed vector space and let $f : V \to V'$ be a continuous mapping with the following property: There exists $r > 0$ such that

$$\langle f(v), v\rangle \ge 0 \quad \text{for all } v \in V \text{ such that } \|v\| = r.$$

Then there exists $v_0 \in V$ such that $\|v_0\| \le r$ and $f(v_0) = 0$.

(iv) Show that the sequence $\{u^n\}_{n\in\mathbb{N}}$ admits a subsequence which converges weakly to u which is solution of the weak formulation above, and further $\|u\|_{H_0^1(\Omega)} \le \|l\|_{H^{-1}(\Omega)}$.

References

[Bro65] F. E. Browder. Nonexpansive nonlinear operators in a Banach space. *Proc. Natl. Acad. Sci. USA*, 54:1041–1044, 1965.

[Kor08] A. Korn. Solution générale du problème d'équilibre dans la théorie de l'élasticité dans le cas où les efforts donnés à la surface. *Toulouse Ann. (2)*, 10:165–269, 1908.

[Kor09] A. Korn. Über einige Ungleichungen, welche in der Theorie der elastischen und elektrischen Schwingungen eine Rolle spielen. Krak. Anz., 705-724 (1909)., 1909.

[3] Named after Felix Browder [Bro65]