Exercise Sheet 12

Introduction to Partial Differential Equations (W. S. 2024/25)

EPFL, Mathematics section, Dr. Nicola De Nitti

- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

Exercise 1. Consider the semilinear problem

$$\begin{cases} -\Delta u = |u|^{p-1}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

Prove that, if $n \geq 3$ and $1 < p < \frac{n+2}{n-2}$, then there exists a non-trivial weak solution $u \in H_0^1(\Omega)$ of (1).

Exercise 2. Let Ω be a bounded open subset of $\mathbb{R}^n$, with $n \geq 3$, and let $2 \leq p < 2^* = \frac{2n}{n-2}$. Prove that

$$\inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^p\right)^{2/p}} = \inf_{u \in E} \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x,$$

where

$$E := H_0^1(\Omega) \cap \{u \in L^p(\Omega) : \|u\|_p = 1\},$$

is finite and is a minimum.

Next, consider the nonlinear problem for $2 < p < 2^*$ on Ω bounded:

$$\begin{cases} -\Delta u = M|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Prove that it admits a weak solution $\bar{u} \in H_0^1(\Omega)$ such that $\|\bar{u}\|_p = 1$ and

$$\int_\Omega \nabla \bar{u} \cdot \nabla v - M \int_\Omega |\bar{u}|^{p-2}\bar{u} v = 0 \qquad \text{for all } v \in H_0^1(\Omega).$$

Moreover, prove that

$$M = \int_\Omega |\nabla \bar{u}|^2 = \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^p\right)^{2/p}}.$$

Exercise 3. Prove that, if Ω is an open star-shaped set in $\mathbb{R}^n$. Then $x \cdot \nu(x) \geq 0$ for all $x \in \partial\Omega$ (where $\nu(x)$ denotes the unit outward normal to $\partial\Omega$ at x). We recall that an open set Ω is called *star-shaped with respect to* 0 if, for each $x \in \bar{\Omega}$, the line segment $\{\lambda x : 0 \leq \lambda \leq 1\}$ lies in $\bar{\Omega}$.