Exercise Sheet 12

Introduction to Partial Differential Equations (W. S. 2024/25)

EPFL, Mathematics section, Dr. Nicola De Nitti

- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

Exercise 1. Prove that, if u is a critical point of the functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2} \int_\Omega |Du|^2 \, \mathrm{d}x - \langle f, u \rangle,$$

then u is a weak solution of the homogeneous Dirichlet problem associated with $-\Delta u = f$. Moreover, use Weierstrass theorem to show that there exists a unique $u \in H_0^1(\Omega)$ that minimizes J and, hence, a unique weak solution of the homogeneous Dirichlet problem associated with $-\Delta u = f$.

Exercise 2. Let $\Omega \subset \mathbb{R}^n$ be an open set that is bounded in one direction or has a finite measure. Then in $W_0^{1,p}(\Omega)$ the norm $\|f\|_{W_0^{1,p}(\Omega)} := \|Df\|_{L^p(\Omega)}$ is equivalent to the norm $\|f\|_{W^{1,p}(\Omega)}$, i.e., there exist two constants $C_1, C_2 > 0$ such that

$$C_1 \|f\|_{W_0^{1,p}(\Omega)} \le \|f\|_{W^{1,p}(\Omega)} \le C_2 \|f\|_{W_0^{1,p}(\Omega)}.$$

Moreover, show that $H_0^1(\Omega)$ equipped with the inner product

$$(u, v)_0 = \int_\Omega Du \cdot Dv \, \mathrm{d}x,$$

is a Hilbert space, and the corresponding norm is equivalent to the standard norm on $H_0^1(\Omega)$.

Exercise 3. Do the exercises left during the lecture: that is, Step 2 in the Proof of Poincaré's inequality and the leftover examples in Section 3 of the Lecture Notes.