# Exercise Sheet 11

# Introduction to Partial Differential Equations (W. S. 2024/25)

EPFL, Mathematics section, Dr. Nicola De Nitti

- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

**Exercise 1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and consider the uniformly elliptic operator

$$Lu(x) = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x).$$

Given a continuous function $\phi : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, non-decreasing with respect to the second argument (i.e., $\phi(x,p) \le \phi(x,q)$ for all $x \in \overline{\Omega}$, $p,q \in \mathbb{R}$, $p < q$), we define the semilinear operator

$$[Q(u)](x) = Lu(x) + \phi(x, u(x)).$$

(i) Let $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ be such that we have $Q(u) \le Q(v)$ in $\Omega$ and $u \le v$ on $\partial\Omega$. Show that $u \le v$ in $\Omega$.

Hint: Show that $\Omega' = \{x \in \Omega : \, u(x) > v(x)\}$ is empty by applying a maximum principle for $L$.

(ii) Let $f : \Omega \to \mathbb{R}$. Show that, if the Dirichlet problem

$$\begin{cases} Q(u) = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

admits a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$, then it is unique.

(iii) Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of the Dirichlet problem above. Show an a priori bound of the form

$$\max_{\overline{\Omega}} |u| \le C \left( \sup_{\Omega} |f| + \max_{\overline{\Omega}} |\phi(\cdot, 0)| \right).$$

Hint: Define $\psi(x) = C_0 - C_1|x|^2$, with $C_0$ and $C_1$ chosen in such a way that $\psi \ge 0$ in $\Omega$, $Q(\psi) \ge Q(u)$ in $\Omega$, and $\psi \ge u$ on $\partial\Omega$.

**Exercise 2.** Let $\Omega = B_1(0) \subset \mathbb{R}^n$. Consider $\alpha \in C(\partial\Omega)$, with $\alpha > 0$ on $\partial\Omega$. Also, consider a vector field $\beta : \partial\Omega \to \mathbb{R}^n$, such that $\beta(x) \cdot \nu(x) > 0$ for all $x \in \partial\Omega$, with $\nu(x) = x$ being the usual outward normal vector. Let $u \in C^2(\overline{\Omega})$ satisfy $Lu = 0$ in $\Omega$, with

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i},$$

a uniformly elliptic operator, with $a_{ij}$ and $b_i$ uniformly bounded. Moreover, assume that $u$ satisfies the oblique Robin condition

$$\alpha u + \partial_\beta u := \alpha u + \beta \cdot \nabla u = 0 \quad \text{on } \partial\Omega.$$

Show that $u = 0$ in $\overline{\Omega}$.

Hint 1: Show that $\max_{\overline{\Omega}} u \leq 0 \leq \min_{\overline{\Omega}} u$.

Hint 2: it may help to consider a non-negative auxiliary function $\phi(x) = e^{-\gamma|x|^2} - e^{-\gamma}$, with $\gamma > 0$, on an annulus $\Omega \backslash \overline{B_\rho}(0), \rho < 1$.

**Exercise 3.** Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. Let $u$ be a smooth solution of the uniformly elliptic equation $Lu := -\sum_{i,j-1}^{n} a_{ij}(x) u_{z_i z_j} = 0$ in $\Omega$. Assume that the coefficients have bounded first derivatives.

(i) Set $v := |\nabla u|^2 + \lambda u^2$ and show that

$$Lv \leq 0 \quad \text{in } \Omega$$

if $\lambda$ is large enough.

Hint: Use the fact that $\nabla L u = 0$.

(ii) Deduce

$$\sup_{\Omega} |\nabla u| \leq C \left( \max_{\partial\Omega} |\nabla u| + \max_{\partial\Omega} |u| \right).$$