Exercise Sheet 10

Introduction to Partial Differential Equations (W. S. 2024/25)

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- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

Exercise 1. Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence in a Banach space X. Show

(i) $x_k \to x$ in X implies $x_k \rightharpoonup x$ in X.

(ii) Prove that there exist sequences that converge weakly to the zero function even though all their terms belong to the unit sphere. Conclude that $x_k \rightharpoonup x$ in X does not imply $x_k \to x$ in X.

Hint: Consider the space $X = L^2(0, 2\pi)$ and as a sequence a well-known complete basis.

Exercise 2. Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. Let $(u_m)_{m\in\mathbb{N}}$ be a bounded sequence in $H^1(\Omega)$. Show that there exists a subsequence $(u_{m_k})_{k\in\mathbb{N}}$ and an element $u \in H^1(\Omega)$ such that, as $k \to \infty$,

$$\begin{aligned} u_{m_k} &\to u \quad \text{in } L^2(\Omega), \\ u_{m_k} &\rightharpoonup u \quad \text{in } H^1(\Omega). \end{aligned}$$

Hint: Recall (without proof) the following result: Let X be a Hilbert space. Suppose that the sequence $(u_m)_{m\in\mathbb{N}} \subset X$ is bounded. Then, there exists a subsequence $(u_{m_k})_{k\in\mathbb{N}}$ of $(u_m)_{m\in\mathbb{N}}$ and $u \in X$ such that

$$u_{m_k} \rightharpoonup u \quad \text{in } X.$$

Exercise 3. Let $\Omega = B_1 \subset \mathbb{R}^n$, $n \geq 1$, and define $\Omega_L = \{x \in \Omega : x_1 < 0\}, \Omega_R = \{x \in \Omega : x_1 > 0\}$, and $\Omega_0 = \{x \in \Omega : x_1 = 0\}$. Consider $u_L \in C^1(\bar{\Omega}_L)$ and $u_R \in C^1(\bar{\Omega}_R)$, such that $u_L = u_R$ on Ω_0. Show that $u = u_L \mathbb{1}_{\Omega_L} + u_R \mathbb{1}_{\Omega_n} \in W^{1,p}(\Omega)$ for all $1 \leq p \leq \infty$.

Exercise 4. Given $W^{k,p}(\Omega)$, with $k \geq 0$ and $1 \leq p < \infty$, we define the set $W_0^{k,p}(\Omega) \subset W^{k,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the $W^{k,p}(\Omega)$-topology, i.e. $u \in W_0^{k,p}(\Omega)$ iff there exists a sequence $(\phi_m)_{m\geq 1} \subset C_0^\infty(\Omega)$ such that

$$\lim_{m\to\infty} \|\phi_m - u\|_{W^{k,p}(\Omega)} = \lim_{m\to\infty} \sum_{|\alpha|\leq k} \|D^\alpha (\phi_m - u)\|_{L^p(\Omega)} = 0.$$

Let us define the *zero-extension (linear) operator* $\zeta : L^P(\Omega) \to L^P(\mathbb{R}^n)$ as

$$\zeta : u \mapsto \zeta u = \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^n \backslash \Omega \end{cases}$$

We have the following property (proof omitted): Given $u \in L^p(\Omega)$, if $\zeta u \in W^{k,p}(\mathbb{R}^n)$, then $u \in W_0^{k,p}(\Omega)$.

Let $\Omega \subset \mathbb{R}^n$ be an open domain and $1 \leq p < \infty$, and consider the zero-extension operator ζ defined above.

(i) Show that $\zeta\left(W_0^{k,p}(\Omega)\right) \subset W^{k,p}(\mathbb{R}^n)$, i.e. that, for all $u \in W_0^{k-p}(\Omega)$, we have $\zeta u \in W^{k,p}(\mathbb{R}^n)$.

Hint: show that $D\zeta u = \zeta(Du)$. a.e. in $\mathbb{R}^n$.

(ii) Let Ω be smooth, such that $\mathbb{R}^n \backslash \bar{\Omega} \neq 0$. For which values of k do we have $\zeta\left(W^{k,p}(\Omega)\right) \subset W^{k,p}(\mathbb{R}^{\mathbf{n}})$?

Exercise 5. Let $\Omega = [-1, 1]$ and $1 \leq p < \infty$. Show that $W_0^{2,p}(\Omega) \subsetneq \left(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\right)$.

<u>Hint:</u> You may use the following result (without proof): $u \in W_0^{1,p}([-1, 1])$ if and only if the zero-extension ζu of u to $\mathbb{R}$ satisfies $\zeta u \in W^{1,p}(\mathbb{R})$ for $1 \leq p < \infty$.

Exercise 6. In this exercise, we wish to show that $C^\infty(\bar{\Omega})$ is not always dense in $W^{1,p}(\Omega)$: we need to be very careful about the regularity of the domain.

Let $\mathbb{R}^2 \supset \Omega = B_1 \backslash \{(x, 0), x \geq 0\}$, and consider $u(x, y) = u(\rho, \theta) = 0$ in polar coordinates ($0 < \rho < 1$ and $0 < 0 < 2\pi$).

(i) Show that $u \in W^{1,1}(\Omega)$, but $u \notin W^{1,1}(B_1)$.

(ii) Show that, for a smooth $\phi \in C^\infty\left(\overline{B_1}\right)$, the natural norms in the spaces $W^{1,1}(\Omega)$ and $W^{1,1}(B_1)$ coincide.

(iii) Conclude that is not possible that there exists $\phi_m \in C^\infty\left(\overline{B_1}\right)$ such that $\phi_m|_\Omega \to u$ in $W^{1,1}(\Omega)$.