Exercise Sheet 4

Introduction to Partial Differential Equations (W. S. 2024/25)

EPFL, Mathematics section, Dr. Nicola De Nitti

- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am via email.

Exercise 1. Given a function $u \in C^2(\mathbb{R}^n)$, prove Bochner's identity:[1]

$$\Delta \frac{1}{2}|\nabla u|^2 = \nabla u \cdot \nabla \Delta u + |D^2 u|^2.$$

(In particular, if u is harmonic, then $\Delta \frac{1}{2}|\nabla u|^2 = |D^2 u|^2$.) Use it to prove that an harmonic function u on $\mathbb{R}^n$, with $\int_{\mathbb{R}^n} |\nabla u|^2 \, \mathrm{d}x < \infty$, is constant.

Hint: Consider the quantity $\frac{1}{2}\int_{\mathbb{R}^n} \varphi_R \Delta |\nabla u|^2 \, \mathrm{d}x$, where $\varphi_R(x) := \varphi(x/R)$ and φ is a non-negative smooth function φ such that

$$\varphi(x) = \begin{cases} 1 & \text{in } B_1(0) \\ 0 & \text{in } \mathbb{R}^n \setminus B_2(0) \end{cases}$$

and $|\Delta \varphi| \leq M$ for some $M > 0$.

Exercise 2. Consider a sequence of non-negative harmonic functions $\{u_i\}_{i \in \mathbb{N}}$, defined on a domain Ω in $\mathbb{R}^N$. Use Harnack's inequality to show that, if $\sum_{i=0}^{\infty} u_i$ converges at some $x_0 \in \Omega$, then it converges uniformly on any compact set $K \subset \Omega$. Deduce that the sum U of the series is non-negative and harmonic everywhere on Ω.

Exercise 3. Compute the limit in $\mathcal{D}'(\mathbb{R})$ of the following sequences of distributions.

(i) T_{f_n}, where $f_n(x) = \frac{n}{2}\chi_{[-\frac{1}{n},\frac{1}{n}]}$.

(ii) T_{g_n}, where $g_n(x) = \sin(nx)$.

(iii) $D_n = n\left(\delta_{\frac{1}{n}} - \delta_0\right)$.

(iv) T_{f_n}, where $f_n(x) = \frac{1}{|x| + \frac{1}{n}}$.

[1] Named after Salomon Bochner.

Exercise 4. Compute, using the definition, the distributional derivative of the following distributions in $\mathcal{D}'(\mathbb{R})$.

(i) $T = |x|$.

(ii) $T = H(x)$, where $H(x) = \chi_{[0,+\infty)}$ is the Heaviside function.

(iii) $T = \chi_{[-a,a]}$ for some $a > 0$.