Exercise Sheet 1

Introduction to Partial Differential Equations (W. S. 2024/25)

EPFL, Mathematics section, Dr. Nicola De Nitti

- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am, via moodle.

Exercise 1. Determine whether the following functions are harmonic in their domains of definition:

$$\begin{aligned} u(x,y) &= e^x \cos(y) \\ v(x,y) &= e^x \sin(y) \\ w(x,y) &= \ln\left(x^2+y^2\right) \end{aligned}$$

Exercise 2. Let Ω be an open bounded domain of $\mathbb{R}^d$ with a smooth boundary. Let $u : \bar{\Omega} \to \mathbb{R}$ and $\mathbf{v} : \bar{\Omega} \to \mathbb{R}^d$ be C^1-functions.

(i) Prove the integration by parts formula

$$\int_\Omega u \operatorname{div}(\mathbf{v}) = \int_{\partial\Omega} u\mathbf{v} \cdot \nu - \int_\Omega \nabla u \cdot \mathbf{v},$$

where ν denotes the exterior normal of $\partial\Omega$.

(ii) Let $u, w \in C^2(\bar{\Omega})$. Prove that

$$\begin{aligned} \int_\Omega u\Delta w &= -\int_\Omega \nabla u \cdot \nabla w + \int_{\partial\Omega} u\partial_\nu w \\ &= \int_\Omega w\Delta u + \int_{\partial\Omega} (u\partial_\nu w - w\partial_\nu u) \end{aligned}$$

(iii) Let $v \in C^2(\bar{\Omega})$ be harmonic. Prove that

$$\int_{\partial\Omega} \partial_\nu v = 0$$

Exercise 3. Let Ω be an open bounded connected domain of $\mathbb{R}^d$ with a smooth boundary $\partial\Omega$, and $u \in C^2(\bar{\Omega})$ a harmonic function in Ω. Prove that

(i) if $\partial_\nu u = 0$ on $\partial\Omega$ then u is constant.

(ii) if $u = 0$ on $\partial\Omega$ then $u = 0$.

Exercise 4.

(i) Let $A \in \mathbb{R}^{d\times d}$ be a non-singular matrix, and $u : \mathbb{R}^d \to \mathbb{R}$ and $\mathbf{v} : \mathbb{R}^d \to \mathbb{R}^d$ be C^1-functions. Let $\tilde{u}(x) := u(Ax)$ and $\tilde{\mathbf{v}}(x) := \mathbf{v}(Ax)$. Prove that

$$\nabla\tilde{u}(x) = A^\top \nabla u(Ax) \quad \text{and} \quad \nabla \cdot \tilde{\mathbf{v}}(x) = [\nabla \cdot (A\mathbf{v})](Ax).$$

(ii) Let $Q \in \mathbb{R}^{d\times d}$ be an orthogonal matrix and $u : \mathbb{R}^d \to \mathbb{R}$ be a harmonic function. Show that $\tilde{u}(x) := u(Qx)$ is harmonic.