Exercise Sheet 3

Introduction to Partial Differential Equations (W. S. 2024/25)

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- The exercise series are published every Tuesday morning at 8am on the moodle page of the course. The exercises can be handed in until the following Tuesday at 8am, via moodle.

Exercise 1. This exercise explores the link between holomorphic $\mathbb{C} \to \mathbb{C}$ functions[1] and harmonic functions on $\mathbb{R}^2$.

(i) Let $\tilde{D}$ be an open connected subset of $\mathbb{C}$, and let $f : \tilde{D} \to \mathbb{C}$ be holomorphic. Define $D = \left\{(x, y) \in \mathbb{R}^2 : x + iy \in \tilde{D}\right\}$. Show that the functions $u, v : D \to \mathbb{R}$,

$$u(x, y) = \mathrm{Re}(f(x + iy)), \quad v(x, y) = \mathrm{Im}(f(x + iy))$$

are harmonic in D.

(ii) Let $D \subset \mathbb{R}^2$ be a simply connected domain, and let u be (real-valued) harmonic in D. Define $\tilde{D} = \{x + iy \in \mathbb{C} : (x, y) \in D\}$. Show that there exists a second (real-valued) function v, harmonic in D, such that $f : \tilde{D} \to \mathbb{C}$, defined as

$$f(x + iy) = u(x, y) + iv(x, y)$$

is holomorphic in $\tilde{D}$.

(iii) Show that the v in Point (ii) is unique up to a constant.

Hints: For (i): use the Cauchy–Riemann equations. For (ii), you can use the following fact (of which we omit the proof): Let g be a holomorphic function on a simply connected domain $\tilde{D}$; then, there exists a holomorphic function G on $\tilde{D}$ such that $G' = g$.

[1] Short reminder (from Complex Analysis). Let D be an open set in $\mathbb{C}$. A function $f : D \to \mathbb{C}$ is *holomorphic* if it is complex differentiable at every point of D. The existence of a complex derivative in a neighbourhood is a very strong condition: it implies that a holomorphic function is infinitely differentiable and analytic. The Cauchy–Riemann equations (named after Augustin-Louis Cauchy and Bernhard Riemann) provide a necessary and sufficient condition for a complex function $f(x+iy) = f(x, y) = u(x, y) + iv(x, y)$ of a single complex variable $z = x + iy$, (with $x, y \in \mathbb{R}$) to be complex differentiable. We have that f is complex differentiable at $z = x+iy$ if and only if u and v are real differentiable functions and the partial derivatives of u and v satisfy

$$\partial_x u(x, y) = \partial_y v(x, y) \qquad \text{and} \qquad \partial_y u(x, y) = -\partial_x v(x, y).$$

We note that the term *holomorphic* was introduced in [BB75, §15 fonctions holomorphes].

Exercise 2. Let Ω be open connected and u be harmonic in Ω. Show that if $|u|$ attains its maximum in Ω, then u is constant.

Exercise 3. Let $\Omega = \mathbb{R}^n \backslash \overline{B_1(0)}$. Let $u \in C^2(\bar{\Omega})$ be harmonic in Ω, and such that $\lim_{|x|\to\infty} u(x) = 0$. Show that

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|.$$

Exercise 4. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. Let $b \in L^\infty(\Omega)^n$, and $c \in L^\infty(\Omega)$, with $c > 0$ in Ω. Assume that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$-\Delta u + b \cdot \nabla u + cu = 0, \quad x \in \Omega \tag{1}$$

and $u = 0$ on $\partial\Omega$. Show that $u = 0$ in Ω.

Hint: Show that $\max_{\bar{\Omega}} u \leq 0$ and $\min_{\bar{\Omega}} u \geq 0$. Follow the lines of the alternative proof of the maximum principle.

References

[BB75] C.A. Briot and J.-C. Bouquet. *Théorie des fonctions elliptiques.* Gauthier-Villars, 2nd edition, 1875.