# Introduction to Partial Differential Equations

Lecture notes for the course “Introduction to Partial Differential Equations”
(EPFL, Winter Semester 2024/25).
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# Contents

# Preface

These lecture notes accompany the course "Introduction to Partial Differential Equations", held for the Bachelor's Degree in Mathematics at EPFL in the Winter Semester 2024/25.

Let us start with a quotation from [Die82]:

> The theory of partial differential equations has been studied incessantly for more than two centuries. By reason of its permanent symbiosis with almost all parts of physics, as well as its ever closer connections with many other branches of mathematics, it is one of the largest and most diverse regions of present-day mathematics, and the vastness of its bibliography defies the imagination.

For some insight into the vastness of this research area, the interested reader may consult [Nir94; BB98; Kla00; Kla09; Lue82; Die81].

Although the course title may suggest a broad overview, our focus will be specific: we aim to introduce "elliptic partial differential equations", delving into the theory surrounding both classical and generalized (weak) solutions.

The prerequisites for this course are "Analysis I–IV". We also recommend familiarity with "Measure and Integration" and "Functional Analysis I".

The structure of the course and the core of these lecture notes draws heavily from the material prepared by Fabio Nobile for the course conducted in the Winter Semester 2023/24 at EPFL. In addition, we will be significantly informed by several key textbooks: [Hun14, Chapters 1–4], [Eva10, Chapters 1, 2.2, 5–6, 8, 9.4], [Jos07, Chapters 1–4, 10–14], [Joh82, Chapter 4], [Bre13, Chapters 8–9, Appendix], [Bre11, Chapters 8–9], [HL11, Chapters 1–2, 6], [GT01, Chapters 1–4, 7–8], and [FR22, Chapters 1–2]. In total, these lecture notes have no pretense of originality.

### Exam

A detailed list of examinable topics (a subset of the lecture notes and exercise series) will be provided on Moodle.

The exam consists of a 30-minute oral examination at the blackboard. Each student will select two questions: one from each part of the course (A: classical solutions, B: weak solutions). Students will then have an additional 30 minutes to prepare their answers without external material or support before the oral examination begins. It is essential for each student to arrive on time (i.e., 30 minutes prior to their scheduled oral exam). Each student must bring a CAMIPRO card or an ID card. Paper and pen for preparation will be provided.

### Additional resources


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[Bre13] A. Bressan. *Lecture notes on functional analysis. With applications to linear partial differential equations*. Vol. 143. Grad. Stud. Math. Providence, RI: American Mathematical Society (AMS), 2013.

[Bre11] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. New York, NY: Springer, 2011.

[BB98] H. Brézis and F. Browder. "Partial differential equations in the 20th century". In: *Adv. Math.* 135.1 (1998), pp. 76–144.

[Die82] J. Dieudonné. *A panorama of pure mathematics (as seen by N. Bourbaki). Transl. from the French by I. G. Macdonald.* Vol. 97. Pure Appl. Math., Academic Press. Academic Press, New York, NY, 1982.

[Eva10] L. C. Evans. *Partial differential equations.* 2nd ed. Vol. 19. Grad. Stud. Math. Providence, RI: American Mathematical Society (AMS), 2010.

[FR22] X. Fernández-Real and X. Ros-Oton. *Regularity theory for elliptic PDE.* Zur. Lect. Adv. Math. Berlin: European Mathematical Society (EMS), 2022.

[GT01] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Reprint of the 1998 ed. Class. Math. Berlin: Springer, 2001.

[HL11] Q. Han and F. Lin. *Elliptic partial differential equations.* 2nd ed. Vol. 1. Courant Lect. Notes Math. New York, NY: Courant Institute of Mathematical Sciences; Providence, RI: American Mathematical Society (AMS), 2011.

[Hun14] J. K. Hunter. *Notes on partial differential equations.* 2014. URL: http://www.math.ucdavis.edu/~hunter/pdes/pdes.html.

[Joh82] F. John. *Partial differential equations. 4th ed.* Vol. 1. Appl. Math. Sci. Springer, Cham, 1982.

[Jos07] J. Jost. *Partial differential equations.* 2nd ed. Vol. 214. Grad. Texts Math. New York, NY: Springer, 2007.

[Kla00] S. Klainerman. “PDE as a unified subject”. In: *GAFA 2000. Visions in mathematics—Towards 2000. Proceedings of a meeting, Tel Aviv, Israel, August 25–September 3, 1999. Part I.* Special volume of the journal Geometric and Functional Analysis. Basel: Birkhäuser, 2000, pp. 279–315.

[Nir94] L. Nirenberg. “Partial differential equations in the first half of the century”. In: *Development of mathematics 1900-1950. Based on a symposium organized by the Luxembourg Mathematical Society in June 1992, at Château Bourglinster, Luxembourg.* Basel: Birkhäuser, 1994, pp. 479–515.

CHAPTER 1

# Introduction

## 1. Notation for partial derivatives

Lect. 1, 10.09

Let $\Omega \subset \mathbb{R}^n, n \geqslant 2$, be open, $x = (x_1, \ldots, x_n) \in \Omega$ and $u : \Omega \to \mathbb{R}$ be a scalar function.

The *partial derivatives* of $u$ at $x$, are defined as

$$\frac{\partial u}{\partial x_i}(x) := \lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h} \qquad \text{(if the limit exists)},$$

for $i = 1, \ldots, n$, where $e_i$ denotes the $i$-th standard basis vector of $\mathbb{R}^n$. Commonly used are also the notations $\frac{\partial u}{\partial x_i} = \partial_{x_i} u = u_{x_i}$.

The *partial derivatives of second order* are defined as

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) := \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_j}\right)(x) \qquad \text{(if they exist)}$$

for $i, j = 1, \ldots, n$. Commonly used are also the notations $\frac{\partial^2 u}{\partial x_i \partial x_j} = \partial^2_{x_i x_j} u = u_{x_i x_j}$.

More generally, let $\alpha = (\alpha_1, \ldots \alpha_n) \in \mathbb{N}_0^n$ be a multi-index. Its order is defined as

$$|\alpha| := \sum_{i=1}^n \alpha_i$$

and the corresponding $|\alpha|$*-th order partial derivatives* of $u$ are

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(x) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u(x) \qquad \text{(if they exist)}.$$

Moreover, for $k \in \mathbb{N}$ we denote by

$$D^k u(x) := \{D^\alpha u(x) : |\alpha| = k\}$$

the collection of all $k$-th order partial derivatives of $u$ in $x$.

In particular, we write $D^1 u(x)$ as a column vector (the *gradient* of $u$ at $x$),

$$D^1 u(x) = Du(x) = \begin{pmatrix} \partial_{x_1} u(x) \\ \vdots \\ \partial_{x_n} u(x) \end{pmatrix} = \nabla u(x),$$

and $D^2 u(x)$ as a matrix (the *Hessian* of $u$ at $x$),

$$D^2 u(x) = \begin{pmatrix} \partial^2_{x_1 x_1} u(x) & \ldots & \partial^2_{x_1 x_n} u(x) \\ \vdots & \ddots & \vdots \\ \partial^2_{x_n x_1} u(x) & \ldots & \partial^2_{x_n x_n} u(x) \end{pmatrix}.$$

## 2. What is a partial differential equation?

A *partial differential equation (PDE)* is an equation for an unknown function $u$ of several variables that involves partial derivatives of $u$. The order of the highest partial derivative is called the *order* of the PDE.

Definition 1.1 ($k$-th order PDE). *Let $\Omega \subset \mathbb{R}^n$ be open, $n \geqslant 2$ and $k \in \mathbb{N}$. An expression of the form*

$$F\left(D^k u(x), D^{k-1} u(x), \ldots, u(x), x\right) = 0, \quad x \in \Omega, \tag{2.1}$$

*is called a* $k$-th order PDE, *where*

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \ldots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

*is a given function and* $u : \Omega \to \mathbb{R}$ *is the unknown.*

Definition 1.2 (Classical solution). *A* classical solution *of the PDE* (2.1) *is a $k$-times continuously differentiable function* $u : \Omega \to \mathbb{R}$ *that satisfies* (2.1).

Depending on the structure of the function $F$ in (2.1), we classify PDEs as follows.

Definition 1.3. *The PDE* (2.1) *is* linear *if the function $F$ is linear in $u$ and its derivatives, i.e. if it is of the form*

$$\sum_{|\alpha|\leqslant k} a_\alpha(x) D^\alpha u(x) + f(x) = 0,$$

*for given functions $a_\alpha$ and $f$. Moreover, if $f \equiv 0$, the PDE is called homogeneous and otherwise inhomogeneous.*

*The PDE* (2.1) *is* semilinear *if it is linear in the highest order derivatives, i.e. if it is of the form*

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + a_0\left(D^{k-1}u(x), \ldots, u(x), x\right) = 0,$$

*for given functions $a_\alpha$ and $a_0$.*

*The PDE* (2.1) *is* quasilinear *if it is of the form*

$$\sum_{|\alpha|=k} a_\alpha\left(D^{k-1}u(x), \ldots, u(x), x\right) D^\alpha u(x) + a_0\left(D^{k-1}u(x), \ldots, u(x), x\right) = 0,$$

*for given functions $a_\alpha$ and $a_0$.*

*The PDE* (2.1) *is* fully nonlinear *if $F$ is a nonlinear function of the highest order derivatives $D^k u$.*

For linear homogeneous equations the superposition principle holds, i.e., if $u$ and $v$ are both solutions of the PDE, then the same applies to $\alpha u + \beta v$, for all $\alpha, \beta \in \mathbb{R}$. More generally, if $u_1, \ldots, u_m$ are solutions, then so is any linear combination of these solutions.

Typically, the difficulty of the analysis of a PDE increases with the degree of nonlinearity.

Instead of scalar equations we can also look at systems of PDEs which arise in many applications. Here, several unknown functions $u_1, \ldots, u_m$, $m \geqslant 2$, have to be determined that satisfy a system of $m$ PDEs.

Definition 1.4 ($k$-th order system of PDEs). *An expression of the form* (2.1) *is called a $k$-th* order system of PDEs *if $m \geqslant 2$ and*

$$F : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \ldots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m,$$

*where $u = (u_1, \ldots, u_m) : \Omega \to \mathbb{R}^m$ is the unknown. Here, $D^\alpha u = (D^\alpha u_1, \ldots, D^\alpha u_m)$ and $D^k u = \{D^\alpha u : |\alpha| \leqslant k\}$.*

Definition 1.5 (Classical solution). *A* classical solution *of the system of PDEs* (2.1) *is a $k$-times continuously differentiable function $u : \Omega \to \mathbb{R}^m$ that satisfies* (2.1).

## 3. Type classification of linear second order PDEs

In this course, we mainly focus on linear, scalar PDEs of second order, i.e., equations of the form

$$\sum_{i,j=1}^{n} a_{ij}(x)\partial^2_{x_i x_j} u(x) + \sum_{i=1}^{n} a_i(x)\partial_{x_i} u(x) + a_0(x)u(x) = f(x), \qquad x \in \Omega, \tag{3.1}$$

that we now further classify.

By Schwarz' theorem, the Hessian matrix is symmetric if $u$ is twice continuously differentiable; hence, when working in this regularity class, we can assume that

$$a_{ij} = a_{ji}, \qquad \text{for all} \quad i, j = 1, \ldots, n,$$

i.e., that the coefficients $a_{ij}$ form a symmetric matrix

$$A(x) = \begin{pmatrix} a_{11}(x) & \ldots & a_{1n}(x) \\ \vdots & \ddots & \vdots \\ a_{n1}(x) & \ldots & a_{nn}(x) \end{pmatrix}, \qquad x \in \Omega$$

A useful type classification of the PDE (3.1) is based on the definiteness properties of $A$.

DEFINITION 1.6 (Elliptic, parabolic, and hyperbolic PDEs). *We call the linear second-order PDE* (3.1) elliptic *if $A(x)$ is positive or negative definite,* parabolic *if $A(x)$ is singular (i.e., $\det A(x) = 0$), and* hyperbolic *if one eigenvalue of $A(x)$ has a different sign than all the others (where the eigenvalues are counted according to their multiplicity).*

The following three examples are the archetypes of second-order linear PDEs.

EXAMPLE 1.1 (Laplace equation).

$$\Delta u = \partial^2_{x_1 x_1} u + \ldots + \partial^2_{x_n x_n} u = 0 \qquad \text{in } \Omega,$$

*where $\Omega \subset \mathbb{R}^n$ is open, $u : \Omega \to \mathbb{R}$ and $\Delta$ is the* Laplace operator *or* Laplacian.

*We have $A(x) = \mathbb{I} \in \mathbb{R}^{n\times n}$ (*identity matrix*) and, thus, the PDE is elliptic.*

EXAMPLE 1.2 (Heat equation).

$$\partial_t u - \Delta u = 0 \qquad \text{in } \Omega = I \times U,$$

*where $t \in I$ denotes time, $x \in U$ space, $I \subset \mathbb{R}$ is an open interval, $U \subset \mathbb{R}^n$ is open, $u : I \times U \to \mathbb{R}$, and $\Delta u = \Delta_x u$ is the Laplace operator with respect to $x$.*

*We obtain a singular matrix*

$$A(t,x) = \begin{pmatrix} 0 & 0 \\ 0 & -\mathbb{I} \end{pmatrix},$$

*$\mathbb{I} \in \mathbb{R}^{n\times n}$, and, thus, the PDE is parabolic.*

EXAMPLE 1.3 (Wave equation).

$$\partial_t^2 u - \Delta u = 0 \qquad \text{in } \Omega = I \times U,$$

*where we use the same notation as for the heat equation.*

*In this case, we have*

$$A(t,x) = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{I} \end{pmatrix},$$

*and thus, the PDE hyperbolic.*

## 4. Studying PDEs

A *classical solution* of a $k$-th order PDE is a $k$-times continuously differentiable function that satisfies the PDE pointwise in $\Omega \subset \mathbb{R}^n$.

Often, a PDE possesses families of solutions, but the solution $u$ is uniquely determined if values of $u$ and/or its derivatives are specified on the boundary $\partial\Omega$ of $\Omega$. A PDE together with these boundary conditions is called a *boundary-value problem*. In applications that involve time (see, e.g., Examples 1.2–1.3), we typically consider sets if the form $\Omega := I \times U$ where $I := (t_0, t_1) \subset \mathbb{R}$ is an open interval and $U \subset \mathbb{R}^n$ is open. In this special case, the values of $u$ and/or its derivatives specified at the initial time $t_0$ are called *initial conditions* and the values specified on $\partial U$ *boundary conditions*. A PDE together with initial and boundary conditions is is called an *initial-boundary-value problem*.

In the ideal case, we find explicit solutions for a given PDE, but this is only possible in few particularly simple cases. This classical approach to PDEs that dominated the 19th century was to develop methods for deriving explicit representation formulas for solutions.

If such formulas cannot be found, we aim at proving the existence and studying qualitative properties of solutions. In particular, we say that a problem is *well-posed*[1] if the following properties hold:

(1) there exists a solution;
(2) the solution is unique;
(3) the solution depends continuously on the given data (e.g., parameters, boundary or initial values).

[1] Or *well-posed in the sense of Hadamard*, after [Had02].

For many PDEs the notion of classical solutions is too restrictive and such solutions do not exist. However, one can weaken the concept of solutions and consider so-called weak solutions or distributional solutions which are less regular and satisfy the PDE in a generalized sense. For instance, PDEs describing the occurrence of shocks (essentially, the appearance of discontinuities in the derivatives), require this notion. Moreover, even if classical solutions exist, it is often easier to prove the existence of weak solutions first and then to show that the solutions have a higher regularity and are, in fact, classical solutions of the problem.

## 5. Objectives of this course

In this course, we will focus on elliptic partial differential equations. In particular, we will cover the following topics:

I. Laplacian operator; Laplace equation; mean-value property; maximum principles; Harnack's inequality; Weyl's lemma; fundamental solution; Green function and solutions to the Dirichlet problem for the Laplace equation; Newtonian potential.

II. Theory of distributions; Sobolev spaces; weak derivatives and their properties; density results; extension results; traces; embedding theorems; Poincaré inequalities.

III. Weak solutions of several elliptic PDEs involving the Laplace operator via variational methods.

IV. General second-order linear elliptic PDEs: classical solutions (maximum principles, a priori bounds) and weak solutions theory.

CHAPTER 2

# Laplace equation and harmonic functions

## 1. Harmonic functions

Let $\Omega \subset \mathbb{R}^n$ (with $n \geqslant 1$) be an open set. The first equation we will study is the *Laplace equation*:[1]

$$-\Delta u(x) = 0, \qquad x \in \Omega. \tag{1.1}$$

Remark 2.1. *We recall that*

$$\Delta u(x) = \operatorname{div} \nabla u(x) = \partial^2_{x_1 x_1} u + \ldots + \partial^2_{x_n x_n} u.$$

Definition 2.1 (Harmonic, sub-harmonic, and super-harmonic functions). *A function $u \in C^2(\Omega)$ is called*

*(1) harmonic in $\Omega$ if it satisfies $\Delta u(x) = 0$ for all $x \in \Omega$;*
*(2) sub-harmonic in $\Omega$ if it satisfies $-\Delta u(x) \leqslant 0$ for all $x \in \Omega$;*
*(3) super-harmonic in $\Omega$ if it satisfies $-\Delta u(x) \geqslant 0$, for all $x \in \Omega$.*

As we will see, to recover a unique solution to this equation, suitable boundary conditions should be provided. The most common are the following.

**Dirichlet[2] problem:**

$$\begin{cases} -\Delta u(x) = 0, & x \in \Omega \\ u(x) = g(x), & x \in \partial\Omega \end{cases}$$

**Neumann[3] problem:**

$$\begin{cases} -\Delta u(x) = 0, & x \in \Omega \\ \partial_\nu u(x) = h(x), & x \in \partial\Omega \end{cases}$$

**Robin[4] problem:**

$$\begin{cases} -\Delta u(x) = 0, & x \in \Omega \\ \partial_\nu u(x) + \alpha\, u(x) = h(x), & x \in \partial\Omega \end{cases} \qquad \text{(with } \alpha > 0 \text{ given)}$$

The non homogeneous version of (1.1) is called the *Poisson[5] equation*:[6]

$$-\Delta u(x) = f(x), \qquad x \in \Omega \tag{1.2}$$

for some given continuous function $f : \Omega \to \mathbb{R}^n$.

The Poisson equation appears in many different fields of physics. For instance, it may describe the distribution of temperature in a region $\Omega$, in the presence of a heat source $f$, knowing the temperature at the boundary (Dirichlet problem) or the heat flux through the boundary (Neumann problem).

Example 2.1. *Any affine linear function in $\mathbb{R}^n$ is harmonic in $\mathbb{R}^n$ since*

$$\Delta\left(a + \sum_j b_j x_j\right) = 0 \quad \text{in } \mathbb{R}^n.$$

[1] Named after Pierre-Simon Laplace.
[5] Named after Siméon Denis Poisson
[6] The minus sign in the above equations could, of course, be removed (upon changing the sign of $f$). However, we prefer to keep it and always think of the operator in the Laplace or Poisson equation as "$-\Delta$".

EXAMPLE 2.2. *The function* $\ln\sqrt{x^2+y^2} = \frac{1}{2}\ln\left(x^2+y^2\right)$ *is harmonic in* $\mathbb{R}^2\backslash\{(0,0)\}$. *Indeed, assuming* $(0,0) \neq (x,y) \in \mathbb{R}^2$, *we compute*

$$\begin{aligned}
\nabla\left(\ln\sqrt{x^2+y^2}\right) &= \nabla\frac{1}{2}\ln\left(x^2+y^2\right) = \frac{(x,y)}{x^2+y^2};\\
\Delta\left(\ln\sqrt{x^2+y^2}\right) &= \operatorname{div}\left(\nabla\left(\frac{1}{2}\ln\left(x^2+y^2\right)\right)\right)\\
&= \operatorname{div}\frac{(x,y)}{x^2+y^2} = \frac{\partial}{\partial x}\frac{x}{x^2+y^2} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2}\\
&= \frac{\left(y^2-x^2\right)+\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\\
&= 0.
\end{aligned}$$

EXAMPLE 2.3. *The function* $\left(x^2+y^2+z^2\right)^{-1/2}$ *is harmonic in* $\mathbb{R}^3\backslash\{(0,0,0)\}$. *Indeed, assuming* $(0,0,0) \neq (x,y,z) \in \mathbb{R}^3$, *we compute*

$$\begin{aligned}
\nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) &= \frac{(-x,-y,-z)}{(x^2+y^2+z^2)^{3/2}};\\
\Delta\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) &= -\operatorname{div}\frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}\\
&= -\frac{y^2+z^2-x^2}{\left(x^2+y^2+z^2\right)^{5/2}} - \frac{x^2+y^2-z^2}{\left(x^2+y^2+z^2\right)^{5/2}} - \frac{x^2+z^2-y^2}{\left(x^2+y^2+z^2\right)^{5/2}}\\
&= 0.
\end{aligned}$$

**Lect. 2, 17.09**

EXAMPLE 2.4. *Let* $n \geqslant 2$. *We compute*

$$\Delta|x|^\alpha = (n\alpha + \alpha(\alpha-2))|x|^{\alpha-2}.$$

*Thus* $|x|^\alpha$ *is harmonic for* $\alpha = 2-n$ *and sub-harmonic for* $\alpha \geqslant 2-n$.

EXAMPLE 2.5 (Harmonic polynomials). *Let us find all harmonic polynomials of degree* $n$ *in two variables,*

$$P_n(x,y) = \sum_{k=0}^{n} c_k x^{n-k} y^k,$$

*that are harmonic. To this end, we compute*

$$\begin{aligned}
\Delta P_n(x,y) &= \sum_{k=0}^{n-2} c_k(n-k)(n-k-1)x^{n-k-2}y^k + \sum_{h=2}^{n} c_h h(h-1)x^{n-h}y^{h-2}\\
&= \sum_{k=0}^{n-2} c_k(n-k)(n-k-1)x^{n-k-2}y^k + \sum_{k=0}^{n-2} c_{k+2}(k+2)(k+1)x^{n-k-2}y^k\\
&= \sum_{k=0}^{n-2}\left[c_k(n-k)(n-k-1) + c_{k+2}(k+2)(k+1)\right]x^{n-k-2}y^k
\end{aligned}$$

*and notice that, for* $P_n$ *to be harmonic, each summand must necessarily have zero coefficient. That is, for any* $k$, *we need*

$$c_k(n-k)(n-k-1) + c_{k+2}(k+2)(k+1) = 0.$$

*This condition yields*

$$c_{k+2} = -\frac{(n-k)(n-k-1)}{(k+2)(k+1)}c_k.$$

*Therefore the even coefficients depend on the choice of $C_c$, while the odd ones depend on $c_1$. Let us consider the former ones:*

$$\begin{aligned} c_2 &= -\frac{n(n-1)}{2\cdot 1}C_c = -\frac{n(n-1)(n-2)!}{2!\cdot(n-2)!}C_c = -\binom{n}{2}C_c \\ c_4 &= -\frac{(n-2)(n-3)}{4\cdot 3}c_2 = \frac{n(n-1)(n-2)(n-3)}{4\cdot 3\cdot 2\cdot 1}C_c = \\ &= \frac{n(n-1)(n-2)(n-3)(n-4)!}{4!\cdot(n-4)!}C_c = \binom{n}{4}C_c. \end{aligned}$$

*By induction, we can prove that*

$$c_{2h} = (-1)^h\binom{n}{2h}C_c, \qquad \text{and, analogously,} \qquad c_{2h+1} = (-1)^h\binom{n}{2h+1}\frac{c_1}{n}.$$

*In conclusion*

$$P_n(x,y) = \sum_{k=0}^{n}\tilde{c}_k\binom{n}{k}x^{n-k}y^k, \qquad \text{with} \qquad \tilde{c}_k = \begin{cases}(-1)^hC_c & \text{for } k=2h\\ (-1)^h\frac{c_1}{n} & \text{for } k=2h+1\end{cases}$$

*and $C_c, c_1$ arbitrary.*

Example 2.6. *Let $u:\Omega\to\mathbb{R}$ be harmonic and positive and $\beta\geqslant 1$. Then $u^\beta$ is sub-harmonic. Indeed, we compute*

$$\begin{aligned} \Delta u^\beta &= \sum_{i=1}^{d}\left(\beta u^{\beta-1}\partial^2_{x_i}u + \beta(\beta-1)u^{\beta-2}\partial_{x_i}u\ \partial_{x_i}u\right) \\ &= \sum_{i=1}^{d}\beta(\beta-1)u^{\beta-2}\partial_{x_i}u\ \partial_{x_i}u = \sum_{i=1}^{d}\beta(\beta-1)u^{\beta-2}(\partial_{x_i}u)^2. \end{aligned}$$

Example 2.7. *From Example 2.6, we note that, if $u\in C^2(\mathbb{R}^n)$ is a positive function and $\beta\in\mathbb{R}$, then*

$$\Delta\left(u^\beta\right) = \beta u^{\beta-1}\Delta u + \beta(\beta-1)u^{\beta-2}|\nabla u|^2.$$

Example 2.8. *Let $u:\Omega\to\mathbb{R}$ be harmonic and positive. Then*

$$\Delta\log u = \sum_{i=1}^{n}\left(\frac{\partial^2_{x_i}u}{u} - \frac{\partial_{x_i}u\ \partial_{x_i}u}{u^2}\right) = -\sum_{i=1}^{n}\frac{\partial_{x_i}u\ \partial_{x_i}u}{u^2} = -\sum_{i=1}^{n}\frac{(\partial_{x_i}u)^2}{u^2}.$$

*Thus, $\log u$ is super-harmonic and $-\log u$ is sub-harmonic.*

Example 2.9. *Let $u:\Omega\to\mathbb{R}$ be harmonic, $f:u(\Omega)\to\mathbb{R}$ be a $C^2$ convex function. Then $f\circ u$ is sub-harmonic. We compute*

$$\begin{aligned} \Delta f(u(x)) &= \sum_{i=1}^{n}\left(f'(u(x))\partial^2_{x_i}u + f''(u(x))\partial_{x_i}u\ \partial_{x_i}u\right) \\ &= \sum_{i=1}^{n}f''(u(x))\left(\partial_{x_i}u\right)^2 \qquad \text{(since } u \text{ is harmonic)} \\ &\geqslant 0 \end{aligned}$$

*since for a convex $C^2$-function $f''\geqslant 0$.*

Remark 2.2 (Chain rule). *From Example 2.9, we note that, if $u\in C^2(\mathbb{R}^n)$ and $\varphi\in C^2(\mathbb{R})$,*

$$\Delta\varphi(u) = \varphi''(u)|\nabla u|^2 + \varphi'(u)\Delta u$$

*(which is just the chain rule). If we assume that $\varphi$ is also convex, then $\varphi''\geqslant 0$, and we obtain*

$$(-\Delta)\varphi(u)\leqslant\varphi'(u)(-\Delta)u.$$

Remark 2.3 (Product rule). *If $u,v\in C^2(\mathbb{R}^n)$, we compute*

$$\Delta(uv) = u\Delta v + 2\,\nabla u\cdot\nabla v + v\Delta u.$$

*In particular, we notice that, if $u$ and $v$ are harmonic, then $uv$ is harmonic if and only if $\nabla u\cdot\nabla v\equiv 0$.*

EXAMPLE 2.10. *If $u, v \in C^2(\Omega)$ with $u$ positive, then*

$$\begin{aligned}\Delta\left(u^v\right) =& vu^{v-1}\Delta u + u^v(\log u)\Delta v + v(v-1)u^{v-2}|\nabla u|^2\\ &+ u^v(\log u)^2|\nabla v|^2 + 2u^{v-1}(1+v\log u)\nabla u\cdot\nabla v.\end{aligned}$$

EXAMPLE 2.11 (Laplacian in polar coordinates). *Let $u \in C^2(\mathbb{R}^2)$ and define a function $U$ by $U(r,\theta) = u(r\cos\theta,\ r\sin\theta)$. We compute*

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}.$$

EXAMPLE 2.12 (Laplacian in spherical coordinates). *Let $u \in C^2(\mathbb{R}^3)$ and define a function $U$ by $U(\rho,\theta,\varphi) = u(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)$. We compute*

$$\Delta u = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial U}{\partial\rho}\right) + \frac{1}{\rho^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial U}{\partial\varphi}\right) + \frac{1}{\rho^2\sin^2\varphi}\frac{\partial^2 U}{\partial\theta^2}.$$

## 2. Properties of harmonic functions

Before addressing the question of the existence and uniqueness of solutions of the above-mentioned boundary value problems, we first focus on some important properties of harmonic functions.

**2.1. Mean-value formula.** In what follows, we will use the notation $\omega_n := |\partial B_1(0)|$ for the surface of the unit sphere in $\mathbb{R}^n$. For instance, $\omega_2 = 2\pi$, $\omega_3 = 4\pi$, and the general expression is $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$. We will use $\alpha_n = |B_1(0)|$ to denote the volume of the unit ball in $\mathbb{R}^n$. We can compute the relationship between $\alpha_n$ and $\omega_n$ as follows:

$$\alpha_n = \int_{B_1(0)} 1\,\mathrm{d}x = \int_0^1 |\partial B_r(0)|\,\mathrm{d}r = \omega_n\int_0^1 r^{n-1}\,\mathrm{d}r = \frac{\omega_n}{n}.$$

Finally, by a simple scaling argument, we can check that $|\partial B_r(x)| = r^{n-1}\omega_n$ and $|B_r(x)| = r^n\alpha_n$, for any $r > 0$ and $x \in \mathbb{R}^n$.

Harmonic functions have the following mean-value property which states that the average value of the function over a ball or sphere is equal to its value at the center.

THEOREM 2.1 (Mean-value formula). *Let $\Omega$ be a domain and $u \in C^2(\Omega)$ a harmonic function. Then, for any closed ball $\overline{B_r(x)} \subset \Omega$, it holds*

$$u(x) = \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)\,\mathrm{d}y = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\,\mathrm{d}S(y). \tag{2.1}$$

Following the proof of Theorem 2.1, we can actually show a mean-value property for sub/super-harmonic functions as well.

THEOREM 2.2 (Mean-value property for sub/super-harmonic functions). *Let $u \in C^2(\Omega)$ be a sub-harmonic (resp., super-harmonic) function. Then, for any closed ball $\overline{B_r(x)} \subset \Omega$ it holds*

$$u(x) \leqslant \text{(resp. } \geqslant\text{)}\ \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)\,\mathrm{d}y,$$

$$u(x) \leqslant \text{(resp. } \geqslant\text{)}\ \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\,\mathrm{d}S(y).$$

It follows from these inequalities in Theorem 2.2 that the value of a sub-harmonic (or super-harmonic) function at the center of a ball is less (or greater) than or equal to the value of a harmonic function with the same values on the boundary. Thus, the graphs of sub-harmonic functions lie below the graphs of harmonic functions and the graphs of super-harmonic functions lie above, which explains the terminology.

EXAMPLE 2.13. *The function $u(x) = |x|^4$ is sub-harmonic in $\mathbb{R}^n$ since $\Delta u = 4(n+2)|x|^2 \geqslant 0$. The function is equal to the constant harmonic function $U(x) = 1$ on the sphere $|x| = 1$, and $u(x) \leqslant U(x)$ when $|x| \leqslant 1$.*

PROOF OF THEOREM 2.1. We divide the proof into two parts.

**Step 1.** We first prove

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y) \, \mathrm{d}S(y).$$

To this end, let us define the function

$$\phi(r) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y) \, \mathrm{d}S(y)$$

Writing $y = x + rz$ with $z = \frac{y-x}{r} \in \partial B_1(0)$, we can recast the integral on the unit sphere

$$\phi(r) = \frac{1}{r^{n-1}\omega_n} \int_{\partial B_1(0)} u(x+rz) r^{n-1} \, \mathrm{d}S(z) = \frac{1}{\omega_n} \int_{\partial B_1(0)} u(x+rz) \, \mathrm{d}S(z)$$

Since $u \in C^1\left(\overline{B_r(x)}\right)$, in particular $u$ and $\nabla u$ are uniformly continuous in $\overline{B_r(x)}$ and

$$\phi(0) = \lim_{r \to 0} \phi(r) = \frac{1}{\omega_n} \int_{\partial B_1(0)} \lim_{r \to 0} u(x+rz) dS(z) = u(x)$$

Moreover,

$$\begin{aligned}
\phi'(r) &= \frac{1}{\omega_n} \int_{\partial B_1(0)} \frac{\mathrm{d}}{\mathrm{d}r} u(x+rz) \, \mathrm{d}S(z) \\
&= \frac{1}{\omega_n} \int_{\partial B_1(0)} \nabla u(x+rz) \cdot z \, \mathrm{d}S(z) \quad (\text{ where } z \text{ is the normal unit vector}) \\
&= \frac{1}{r^{n-1}\omega_n} \int_{\partial B_r(x)} \nabla u(y) \cdot \nu(y) \, \mathrm{d}S(y) \\
&= \frac{1}{r^{n-1}\omega_n} \int_{B_r(x)} \Delta u(y) \, \mathrm{d}y = 0 \quad (\text{since } u \text{ is harmonic}), \qquad (2.2)
\end{aligned}$$

which yields $\phi(r) = \phi(0) = u(x)$.

**Step 2.** To prove the first identity,

$$u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y) \, \mathrm{d}y,$$

we compute

$$\begin{aligned}
\int_{B_r(x)} u(y) dy &= \int_0^r \left( \int_{\partial B_t(x)} u(y) dS(y) \right) \mathrm{d}t \\
&= \int_0^r t^{n-1} \omega_n u(x) \, \mathrm{d}t \\
&= \frac{r^n \omega_n}{n} u(x) = |B_r(x)| \, u(x).
\end{aligned}$$

□

PROOF OF THEOREM 2.2. Compared to the proof of Theorem 2.1, the only difference is in line (2.2). □

The converse of this result is also true, i.e. if a function $u \in C^2(\Omega)$ satisfies the mean-value property, then it is harmonic.

THEOREM 2.3 (Mean-value property implies harmonicity). *If $u \in C^2(\Omega)$ satisfies the mean-value property*

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y) \, \mathrm{d}S(y)$$

*for any closed ball $\overline{B_r(x)} \subset \Omega$, then $u$ is harmonic in $\Omega$.*

PROOF. We argue by contradiction. If $u$ is not harmonic, then there exists $x \in \Omega$ such that $\Delta u(x) \neq 0$. Assume $\Delta u(x) > 0$. Since $u \in C^2(\Omega)$, then there exists $s > 0$ such that $\Delta u(y) > 0$, for all $y \in \overline{B_s(x)}$ and $\overline{B_s(x)} \subset \Omega$. As in the proof of Theorem 2.1, we define the function

$$\phi(r) := \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\,\mathrm{d}S(y), \quad 0 < r \leqslant s,$$

and compute that

$$0 = \phi'(r) = \frac{1}{r^{n-1}\omega_n} \int_{B_{r(x)}} \Delta u(y)\,\mathrm{d}y > 0,$$

a contradiction. □

In Theorem 2.3, we can drop the hypothesis $u \in C^2(\Omega)$ and the claim remains true for any $u \in C^0(\Omega)$, i.e., if a function $u \in C^0(\Omega)$ satisfies the mean-value formula (2.1) for any closed ball contained in $\Omega$, then it is automatically $C^2(\Omega)$ and harmonic. More than that, it is of class $C^\infty(\Omega)$. To prove this result we recall first the notion of mollifier[7].

DEFINITION 2.2 (Standard mollifier). *Let us consider the $C^\infty$ function*

$$\phi : \mathbb{R}_+ \to \mathbb{R}, \quad \phi(r) := \begin{cases} C\exp\left(\frac{1}{r^2-1}\right), & \text{if } 0 \leqslant r < 1, \\ 0, & \text{if } r \geqslant 1, \end{cases}$$

*with $C > 0$ chosen such that $\int_{\mathbb{R}^n} \phi(\|x\|)\,\mathrm{d}x = 1$. For any $\varepsilon > 0$ we call standard $\varepsilon$-mollifier the function*

$$\eta_\varepsilon : \mathbb{R}^n \to \mathbb{R}, \quad \eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\phi\left(\frac{\|x\|}{\varepsilon}\right)$$

We can show that the $\varepsilon$-mollifier satisfies $\eta_\varepsilon \in C^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \eta_\varepsilon(x)\,\mathrm{d}x = 1$, and $\operatorname{supp}(\eta_\varepsilon) = \overline{B_\varepsilon(0)}$ where, for a function $f : \mathbb{R}^n \to \mathbb{R}$, we recall that $\operatorname{supp}(f) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$.

DEFINITION 2.3 (Mollification). *Let $\Omega \subset \mathbb{R}^n$ be a domain and, for all $\varepsilon > 0$, define the subdomain $\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$. For $f : \Omega \to \mathbb{R}$ locally intergrable, we call $\varepsilon$-mollification of $f$ the function $f_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$,*

$$f_\varepsilon(x) = (\eta_\varepsilon * f)(x) = \int_\Omega \eta_\varepsilon(x-y)f(y)\,\mathrm{d}y = \int_{B_\varepsilon(0)} \eta_\varepsilon(y)f(x-y)\,\mathrm{d}y$$

It can be proven that $f_\varepsilon \in C^\infty(\Omega_\varepsilon)$ (for all $\varepsilon > 0$) and that the following properties hold:

(1) $f_\varepsilon \to f$ a.e. in $\Omega$ as $\varepsilon \to 0$;
(2) if $f \in C^0(\Omega)$ then $f^\varepsilon \to f$ uniformly on any compact subset of $\Omega$;
(3) if $f \in L^p_{\mathrm{loc}}(\Omega)$ then $f_\varepsilon \to f$ in $L^p_{\mathrm{loc}}(\Omega)$.

THEOREM 2.4. *If $u \in C^0(\Omega)$ satisfies the mean-value property*

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\,\mathrm{d}S(y)$$

*for any closed ball $\overline{B_r(x)} \subset \Omega$, then $u \in C^\infty(\Omega)$ and is harmonic in $\Omega$.*

[7] Also known as *Friedrichs mollifier*, after Kurt Otto Friedrichs [Fri44], or *approximations of the identity*

PROOF. Let us consider an $\varepsilon$-mollification of $u$, $u_\varepsilon = \eta_\varepsilon * u : \Omega_\varepsilon \to \mathbb{R}$ and recall that $\eta_\varepsilon$ is a radial function, i.e. $\eta_\varepsilon(x) = \varepsilon^{-n}\phi(\|x\|/\varepsilon)$ and $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$ for all $\varepsilon > 0$. Then

$$\begin{aligned}
u_\varepsilon(x) &= \int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y = \int_{B_\varepsilon(x)} \frac{1}{\varepsilon^n}\phi\left(\frac{\|x-y\|}{\varepsilon}\right)u(y)\,\mathrm{d}y \\
&= \int_0^\varepsilon \left(\int_{\partial B_t(x)} \frac{1}{\varepsilon^n}\phi\left(\frac{t}{\varepsilon}\right)u(y)\,\mathrm{d}S(y)\right)\mathrm{d}t \\
&= \frac{1}{\varepsilon^n}\int_0^\varepsilon \phi\left(\frac{t}{\varepsilon}\right)\left(\int_{\partial B_t(x)} u(y)\,\mathrm{d}S(y)\right)\mathrm{d}t \\
&= \frac{1}{\varepsilon^n}\int_0^\varepsilon \phi\left(\frac{t}{\varepsilon}\right)|\partial B_t(x)|\,u(x)\,\mathrm{d}t \qquad \text{(by mean-value property)} \\
&= \frac{u(x)}{\varepsilon^n}\int_0^\varepsilon \phi\left(\frac{t}{\varepsilon}\right)|\partial B_t(x)|\,\mathrm{d}t \\
&= u(x)\int_{B_\varepsilon(0)} \eta_\varepsilon(y)\,\mathrm{d}y \\
&= u(x).
\end{aligned}$$

Hence $u(x) = u_\varepsilon(x)$ for all $x \in \Omega_\varepsilon$ and we conclude that $u \in C^\infty(\Omega_\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, it follows that $u \in C^\infty(\Omega)$, hence, in particular, $u \in C^2(\Omega)$ and, by Theorem 2.3, $u$ is harmonic in $\Omega$. ☐

REMARK 2.4. *The stronger result of Theorem 2.4 does not extend to sub/super harmonic functions, in general. If a function $u \in C^0(\Omega)$ satisfies $u(x) \leqslant \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)\,\mathrm{d}y$ for any closed $\overline{B_r(x)} \subset \Omega$, it is not true, in general, that $u \in C^\infty(\Omega)$, nor that $-\Delta u \leqslant 0$ (the Laplacian of $u$ might not even exist).*

We stress that the $C^\infty$ regularity result for harmonic functions contained in Theorem 2.4 says nothing about the behavior of $u$ at the boundary of $\Omega$.

EXAMPLE 2.14. *We can check that the functions*

$$u(x,y) = \frac{x}{x^2+y^2}, \quad v(x,y) = -\frac{y}{x^2+y^2}$$

*are harmonic and $C^\infty$ in the open unit disc*

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 < 1\right\}.$$

*However, both are unbounded as $(x,y) \to (0,0) \in \partial\Omega$.*

**2.2. Derivative estimates.** An important feature of the Laplace equation is that we can estimate the derivatives of a solution in a ball in terms of the solution on a larger ball.

THEOREM 2.5 (Gradient estimate). *Let us suppose that $u \in C^2(\Omega)$ is harmonic in the open set $\Omega$ and $B_r(x) \Subset \Omega$. Then, for any $1 \leqslant i \leqslant n$,*

$$|\partial_i u(x)| \leqslant \frac{n}{r}\max_{x\in\overline{B_r}(x)} |u(x)|.$$

PROOF. Since $u$ is smooth, differentiation of Laplace's equation with respect to $x_i$ shows that $\partial_i u$ is harmonic. Therefore, by the mean-value property for balls and the divergence theorem,

$$\partial_i u = \fint_{B_{r(x)}} \partial_i u(x)\,\mathrm{d}x = \frac{1}{\alpha_n r^n}\int_{\partial B_r(x)} u(y)\,\nu_i(y)\,\mathrm{d}S(y).$$

Taking the absolute value of this equation and using the estimate

$$\left|\int_{\partial B_r(x)} u\nu_i\,\mathrm{d}S\right| \leqslant n\alpha_n r^{n-1}\max_{B_r(x)} |u|$$

we get the result. ☐

One consequence of Theorem 2.5 is that a bounded harmonic function on $\mathbb{R}^n$ is constant.[8]

COROLLARY 2.1 (Liouville's theorem). *If $u \in C^2(\mathbb{R}^n)$ is bounded and harmonic in $\mathbb{R}^n$, then $u$ is constant.*

PROOF. If $|u| \leqslant M$ on $\mathbb{R}^n$, then Theorem 2.5 implies that

$$|\partial_i u(x)| \leqslant \frac{Mn}{r}$$

for any $r > 0$. Taking the limit as $r \to \infty$, we conclude that $\nabla u = 0$, so $u$ is constant. □

We will also present an alternative proof of Corollary 2.1, due to [Nel61].

ALTERNATIVE PROOF OF COROLLARY 2.1. Let $r > 0$ and consider $x, y \in \mathbb{R}^n$. The mean-value property yields:

$$\begin{aligned} |u(x) - u(y)| &= \frac{1}{|B_r|} \left| \int_{B_r(x)} u(z)\,\mathrm{d}z - \int_{B_r(y)} u(z)\,\mathrm{d}z \right| \\ &\leqslant M \frac{|B_r(x) \triangle B_r(y)|}{|B_r|} \\ &\longrightarrow 0 \qquad \text{as } r \to +\infty. \end{aligned}$$

Thus $u(x) = u(y)$ and we conclude that $u$ is a constant from the arbitrariness of $x, y \in \mathbb{R}^n$. □

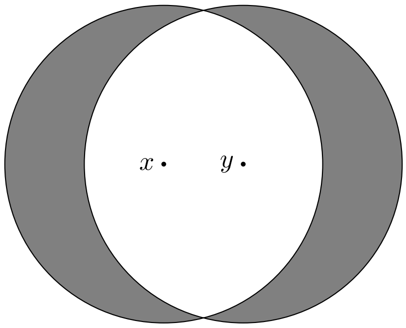


FIGURE 1. Illustration of the argument in the alternative proof of Corollary 2.1.

Lect. 3, 24.09 We can also prove a derivative estimate in terms of the $L^1$-norm of $u$.

THEOREM 2.6. *Let us suppose that $u \in C^2(\Omega)$ is harmonic in the open set $\Omega \subset \mathbb{R}^n$ and $B_r(x) \Subset \Omega$. Then, for any $1 \leqslant j \leqslant n$,*

$$|u(x)| \leqslant \frac{1}{\alpha_n r^n} \|u\|_{L^1(B_r(x))}, \tag{2.3}$$

$$\left|\partial_{x_j} u(x)\right| \leqslant \frac{2^{n+1} n}{\alpha_n r^{n+1}} \|u\|_{L^1(B_r(x))}. \tag{2.4}$$

PROOF. By the mean-value property,

$$|u(x)| = \left| \frac{1}{\alpha_n r^n} \int_{B_r(x)} u(y)\,\mathrm{d}y \right| \leqslant \frac{1}{\alpha_n r^n} \int_{B_r(x)} |u(y)|\,\mathrm{d}y = \frac{1}{\alpha_n r^n} \|u\|_{L^1(B_r(x))},$$

which yields (2.3).

[8] This result is called Liouville's theorem, after Joseph Liouville. Actually, this statement first appeared in [Bôc03] and was later rediscovered in [Pic24]. On the other hand, Liouville, in [Lio80], proved that "A doubly periodic function without poles is identically constant". Finally, the statement of Liouville's theorem about holomorphic functions is actually due to Augustin-Louis Cauchy [Cau44], in response to [Lio80].

To show (2.4), note that $\partial_{x_j} u$ is harmonic. Thus, by the divergence theorem, we have

$$\begin{aligned}
\left|\partial_{x_j} u(x)\right| &= \frac{1}{\alpha_n \left(\frac{r}{2}\right)^n} \left| \int_{B_{r/2}(x)} \partial_{x_j} u(y)\, \mathrm{d}y \right| \\
&= \frac{2^n}{\alpha_n r^n} \left| \int_{\partial B_{r/2}(x)} u(y)\nu_j \, \mathrm{d}S(y) \right| \\
&\leqslant \frac{2^n}{\alpha_n r^n} \int_{\partial B_{r/2}(x)} |u(y)| \, \mathrm{d}S(y).
\end{aligned}$$

Within the integral, we can apply (2.3) over $B_{r/2}(y)$, obtaining

$$\begin{aligned}
\left|\partial_{x_j} u(x)\right| &\leqslant \frac{2^n}{\alpha_n r^n} \int_{\partial B_{r/2}(x)} \frac{2^n}{\alpha_n r^n} \|u\|_{L^1\left(B_{r/2}(y)\right)} \, \mathrm{d}S(y) \\
&= \frac{2^{2n}}{\alpha_n^2 r^{2n}} \int_{\partial B_{r/2}(x)} \|u\|_{L^1\left(B_{r/2}(y)\right)} \, \mathrm{d}S(y).
\end{aligned}$$

Since $B_{r/2}(y) \subset B_r(x)$, we have $\|u\|_{L^1\left(B_{r/2}(y)\right)} \leqslant \|u\|_{L^1(B_r(x))}$, which leads to

$$\left|\partial_{x_j} u(x)\right| \leqslant \frac{2^{2n}}{\alpha_n^2 r^{2n}} \int_{\partial B_{r/2}(x)} \|u\|_{L^1(B_r(x))} \, \mathrm{d}S(y) = \frac{2^{2n}}{\alpha_n^2 r^{2n}} \|u\|_{L^1(B_r(x))} \int_{\partial B_{r/2}(x)} \mathrm{d}y.$$

The claim follows by observing that

$$\int_{\partial B_{r/2}(x)} \mathrm{d}y = \omega_n \left(\frac{r}{2}\right)^{n-1} = n\alpha_n \left(\frac{r}{2}\right)^{n-1}.$$

□

We can extend the estimate in Theorem 2.5 to higher-order derivatives.

THEOREM 2.7 (Estimate on higher derivatives). *Let us suppose that $u \in C^2(\Omega)$ is harmonic in the open set $\Omega$ and $B_r(x) \Subset \Omega$. Then for any multi-index $\alpha \in \mathbb{N}_0^n$ of order $k = |\alpha|$,*

$$|\partial^\alpha u(x)| \leqslant \frac{n^k e^{k-1} k!}{r^k} \max_{\bar{B}_r(x)} |u|.$$

PROOF. We prove the result by induction on $|\alpha| = k$.
Base case: From Theorem 2.5, the result is true when $k = 1$.
Induction step: Let us suppose that the result is true when $|\alpha| = k$. We will prove it for $|\alpha| = k + 1$.

If $|\alpha| = k + 1$, we may write $\partial^\alpha = \partial_{x_i} \partial^\beta$ where $1 \leqslant i \leqslant n$ and $|\beta| = k$. For $0 < \theta < 1$, let $\rho := (1 - \theta)r$. Then, since $\partial^\beta u$ is harmonic and $B_\rho(x) \Subset \Omega$, Theorem 2.5 implies that

$$|\partial^\alpha u(x)| \leqslant \frac{n}{\rho} \max_{x \in \overline{B_r(x)}} |\partial^\beta u(x)|.$$

Let $y \in B_\rho(x)$. Then $B_{r-\rho}(y) \subset B_r(x)$, and, using the induction hypothesis, we get

$$\left|\partial^\beta u(y)\right| \leqslant \frac{n^k e^{k-1} k!}{(r - \rho)^k} \max_{B_{r-\rho}(y)} |u| \leqslant \frac{n^k e^{k-1} k!}{r^k \theta^k} \max_{B_r(x)} |u|.$$

We then deduce that

$$|\partial^\alpha u(x)| \leqslant \frac{n^{k+1} e^{k-1} k!}{r^{k+1} \theta^k (1 - \theta)} \max_{B_r(x)} |u|.$$

Choosing $\theta = k/(k + 1)$ and, using the inequality

$$\frac{1}{\theta^k (1 - \theta)} = \left(1 + \frac{1}{k}\right)^k (k + 1) \leqslant e(k + 1),$$

we get

$$|\partial^\alpha u(x)| \leqslant \frac{n^{k+1} e^k (k + 1)!}{r^{k+1}} \max_{B_r(x)} |u|.$$

□

A consequence of this estimate is that the Taylor series of $u$ converges to $u$ near any point.

THEOREM 2.8 (Harmonic functions are analytic). *If $u \in C^2(\Omega)$ is harmonic in an open set $\Omega$ then $u$ is real-analytic in $\Omega$.*

PROOF. Suppose that $x \in \Omega$ and choose $r > 0$ such that $B_{2r}(x) \Subset \Omega$. Since $u \in C^\infty(\Omega)$, we may expand it in a Taylor series with remainder of any order $k \in \mathbb{N}$ to get

$$u(x+h) = \sum_{|\alpha| \leqslant k-1} \frac{\partial^\alpha u(x)}{\alpha!} h^\alpha + R_k(x,h),$$

where we assume that $|h| < r$.

The remainder[9] is given by

$$R_k(x,h) = \sum_{|\alpha|=k} \frac{\partial^\alpha u(x+\theta h)}{\alpha!} h^\alpha,$$

for some $0 < \theta < 1$. We have to show that $R_k \to 0$ as $k \to +\infty$ (for a sufficiently small $|h|$). To estimate the remainder, we use Theorem 2.7:

$$|\partial^\alpha u(x+\theta h)| \leqslant \frac{n^k e^{k-1} k!}{r^k} \max_{\bar{B}_{r(x+\theta h)}} |u|.$$

Since $|h| < r$, we have $B_r(x+\theta h) \subset B_{2r}(x)$; so, for any $0 < \theta < 1$, we have

$$\max_{B_r(x+\theta h)} |u| \leqslant M, \quad M = \max_{B_{2r}(x)} |u|$$

and then

$$|\partial^\alpha u(x+\theta h)| \leqslant \frac{M n^k e^{k-1} k!}{r^k}$$

Since $|h^\alpha| \leqslant |h|^k$ when $|\alpha| = k$, we deduce

$$|R_k(x,h)| \leqslant \frac{M n^k e^{k-1} |h|^k k!}{r^k} \left( \sum_{|\alpha|=k} \frac{1}{\alpha!} \right)$$

The multinomial expansion

$$n^k = (1+1+\cdots+1)^k = \sum_{|\alpha|=k} \binom{k}{\alpha} = \sum_{|\alpha|=k} \frac{k!}{\alpha!}$$

shows that

$$\sum_{|\alpha|=k} \frac{1}{\alpha!} = \frac{n^k}{k!}.$$

In conclusion, we have

$$|R_k(x,h)| \leqslant \frac{M}{e} \left( \frac{n^2 e |h|}{r} \right)^k$$

and, thus, $R_k(x,h) \to 0$ as $k \to \infty$ provided that $|h| < \frac{r}{n^2 e}$. □

**2.3. Maximum principles.** We present now a second important property of harmonic and sub/super-harmonic functions, the *maximum principle*.

[9] Let us recall Taylor's theorem: If $u \in C^k(B_r(x))$ and $h \in B_r(0)$, then

$$u(x+h) = \sum_{|\alpha| \leqslant k-1} \frac{\partial^\alpha u(x)}{\alpha!} h^\alpha + R_k(x,h),$$

where the remainder is given by

$$R_k(x,h) = \sum_{|\alpha|=k} \frac{\partial^\alpha u(x+\theta h)}{\alpha!} h^\alpha$$

for some $0 < \theta < 1$. We also recall that, for a multi-index $\alpha \in \mathbb{N}_0^n$, the factorial is defined as $\alpha! = \alpha_1! \ldots \alpha_k!$

2.3.1. *Strong maximum principle.* First, we present the *strong maximum principle*, which states that a sub/super-harmonic function which attains an interior maximum/minimum is a trivial constant function.

THEOREM 2.9 (Strong maximum principle for sub-harmonic functions). *Let us suppose that $\Omega \subset \mathbb{R}^n$ is a connected open set and $u \in C^2(\Omega)$. If $u$ is sub-harmonic and attains a global maximum value in $\Omega$, then $u$ is constant in $\Omega$.*

PROOF. By assumption, $u$ is bounded from above and attains its maximum in $\Omega$. Let

$$M := \max_{\Omega} u$$

and consider

$$F := \{x \in \Omega : u(x) = M\}.$$

Then $F$ is non-empty (by assumption, since the maximum is attained) and relatively closed in $\Omega$ (since $u$ is continuous). If $x \in F$ and $B_r(x) \Subset \Omega$, then the mean value inequality for sub-harmonic functions implies that

$$\fint_{B_r(x)} [u(y) - u(x)]\,\mathrm{d}y = \fint_{B_r(x)} u(y)\,\mathrm{d}y - u(x) \geqslant 0.$$

Since $u$ attains its maximum at $x$, we have $u(y) - u(x) \leqslant 0$ for all $y \in \Omega$, and it follows that $u(y) = u(x)$ in $B_r(x)$. Therefore, $F$ is open as well as closed. Since $\Omega$ is connected, and $F$ is non-empty, we must have $F = \Omega$, so $u$ is constant in $\Omega$.

If $\Omega$ is not connected, then $u$ is constant in any connected component of $\Omega$ that contains an interior point where $u$ attains a maximum value.

□

EXAMPLE 2.15. *The function $u(x) = |x|^2$ is sub-harmonic in $\mathbb{R}^n$. It attains a global minimum in $\mathbb{R}^n$ at the origin, but it does not attain a global maximum in any open set $\Omega \subset \mathbb{R}^n$. It does, of course, attain a maximum in any bounded closed set $\bar{\Omega}$, but the attainment of a maximum at a boundary point instead of an interior point does not imply that a sub-harmonic function is constant.*

It follows immediately that super-harmonic functions satisfy a minimum principle, and harmonic functions satisfy a maximum and minimum principle.

THEOREM 2.10 (Strong maximum principle for harmonic functions). *Let us suppose that $\Omega$ is a connected open set and $u \in C^2(\Omega)$. If $u$ is harmonic and attains either a global minimum or maximum in $\Omega$, then $u$ is constant.*

PROOF. Any super-harmonic function $u$ that attains a minimum in $\Omega$ is constant, since $-u$ is sub-harmonic and attains a maximum. A harmonic function is both sub-harmonic and super-harmonic. □

EXAMPLE 2.16. *The function*

$$u(x, y) = x^2 - y^2$$

*is harmonic in $\mathbb{R}^2$. It has a critical point at 0, meaning that $Du(0) = 0$. This critical point is a saddle-point, however, not an extreme value. Note also that*

$$\fint_{B_r(0)} u\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2\pi}\int_0^{2\pi} \left(\cos^2\theta - \sin^2\theta\right)\,\mathrm{d}\theta = 0,$$

*as required by the mean-value property.*

2.3.2. *Weak maximum principle.* Theorem 2.10 leads to a *weak maximum principle* for harmonic functions, which states that the function is bounded inside a domain by its values on the boundary. In physical terms, this means for example that the interior of a bounded region which contains no heat sources or sinks cannot be hotter than the maximum temperature on the boundary or colder than the minimum temperature on the boundary.

THEOREM 2.11 (Weak maximum principle). *Let $\Omega$ be bounded and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with $-\Delta u \leqslant 0$ (resp. $-\Delta u \geqslant 0$ ) in $\Omega$. Then*

$$\max_{x\in\bar{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x) \qquad \left( \textit{resp. } \min_{x\in\bar{\Omega}} u(x) = \min_{x\in\partial\Omega} u(x) \right)$$

*In particular, if $u$ is harmonic in $\Omega$, then*

$$\min_{y\in\partial\Omega} u(y) \leqslant u(x) \leqslant \max_{y\in\partial\Omega} u(y), \quad \forall x \in \Omega$$

PROOF. Suppose $u$ sub-harmonic in $\Omega$ and let $M = \max_{x\in\bar{\Omega}} u(x)$ which is finite since $u \in C^0(\bar{\Omega})$. If there exists $y \in \Omega$ such that $u(y) = M$, then $u$ is constant by strong maximum principle and the thesis follows. If there is no such $y \in \Omega$, then $u(y) < M = \max_{x\in\partial\Omega} u(x)$ for all $y \in \Omega$. □

EXAMPLE 2.17. *The harmonic function $u(x) = x_1$ on the half-space $\{x \in \mathbb{R}^n : x_1 > 0\}$ is equal to zero on the boundary, but is positive in the domain. We cannot apply the maximum principle because the domain is unbounded.*

REMARK 2.5. *The maximum principle is a second-order phenomenon. The function $u : [0,1] \to \mathbb{R}$ defined as $u(x) = 3x^2 - 4x^3$ satisfies $\frac{\mathrm{d}^4}{\mathrm{d}x^4} u(x) = 0$ but $x = 1/2$ is an interior maximum.*

COROLLARY 2.2. *Let $u$ and $v$ be harmonic and sub-harmonic in a bounded domain $\Omega$, respectively. If $u \geqslant v$ on $\partial\Omega$, then $u \geqslant v$ in $\Omega$.*

PROOF. We observe that $v - u$ is sub-harmonic in $\Omega$, and that $v - u \leqslant 0$ on $\partial\Omega$. By the weak maximum principle, $\max_\Omega(v - u) \leqslant 0$, and the claim follows. □

2.3.3. *Alternative proof of the weak maximum principle.* We present also a proof by contradiction, which does not use the strong maximum principle and generalizes easily to more general second-order elliptic equations.

ALTERNATIVE PROOF OF THEOREM 2.11. We will split the proof into two cases.

Case 1: $-\Delta u < 0$ (with strict inequality). Let us consider the case $-\Delta u < 0$. Let us assume, for the sake of finding a contradiction, that

$$\max_{x\in\bar{\Omega}} u(x) > \max_{x\in\partial\Omega} u(x).$$

Then there exists $y \in \Omega$ such that $u(y) = M = \max_{x\in\bar{\Omega}} u(x)$ and, since $u \in C^2(\Omega)$, we have $\partial^2_{x_i} u(y) \leqslant 0$ for $i = 1, \ldots, n$, which contradicts $-\Delta u(y) < 0$.

Case 2: $-\Delta u \leqslant 0$. In this case, we introduce an auxiliary function $v \in C^2(\bar{\Omega})$ which satisfies $-\Delta v(x) \leqslant \alpha < 0$ for all $x \in \Omega$ and some $\alpha > 0$. For instance, we can take

$$v(x) := |x|^2 = \sum_{i=1}^n x_i^2.$$

Then, for any $\varepsilon > 0$, the function $u_\varepsilon := u + \varepsilon v$ satisfies

$$-\Delta u_\varepsilon \leqslant -2n\varepsilon < 0 \qquad \text{in } \Omega$$

and (by the analysis in Case 1)

$$\max_{x\in\bar{\Omega}} u_\varepsilon(x) = \max_{x\in\partial\Omega} u_\varepsilon(x).$$

Since $\Omega$ is bounded, there exists $R > 0$ such that $|x| \leqslant R$ for all $x \in \Omega$. Then

$$\max_{\bar{\Omega}} u \leqslant \max_{\bar{\Omega}} u_\varepsilon \leqslant \max_{\partial\Omega} u_\varepsilon \leqslant \max_{\partial\Omega} u + \varepsilon R^2,$$

and, letting $\varepsilon \searrow 0$, we conclude. Finally, we note that the statement for super-harmonic functions, $-\Delta u \geqslant 0$ is proved in the same way (considering that, if $u$ is sub-harmonic, then $-u$ is super-harmonic).

In particular, for harmonic functions, we conclude

$$\min_{y\in\partial\Omega} u(y) = \min_{y\in\bar{\Omega}} u(y) \leqslant u(x) \leqslant \max_{y\in\bar{\Omega}} u(y) = \max_{y\in\partial\Omega} u(y), \quad \text{for all } x \in \Omega.$$

□

2.3.4. *Application to the study of uniqueness for the Laplace equation.* The weak maximum principle can be used to prove uniqueness of solutions of the Dirichlet problem in bounded domains. Indeed, let us assume that two solutions $u_1$ and $u_2$ exist. Then, $v = u_1 - u_2$ satisfies the problem

$$\begin{cases} -\Delta v = 0, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

By the weak maximum principle, we have $v = 0$, hence $u_1 = u_2$.

The same principle can be used to establish stability estimates for the solution. Indeed, if $u$ is a solution of the Dirichlet problem with boundary data $g \in C^0(\partial\Omega)$, then

$$\min_{y\in\partial\Omega} g(y) \leqslant u(x) \leqslant \max_{y\in\partial\Omega} g(y), \quad \text{for all } x \in \Omega,$$

which implies

$$\|u\|_{C^0(\bar{\Omega})} \leqslant \|g\|_{C^0(\partial\Omega)},$$

i.e., the $C^0$-norm of the solution is controlled by the $C^0$-norm of the data.

Similarly, if $u_{g_1}$ and $u_{g_2}$ solve the problems

$$\begin{cases} -\Delta u_{g_1} = 0, & x \in \Omega, \\ u_{g_1} = g_1, & x \in \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_{g_2} = 0, & x \in \Omega, \\ u_{g_2} = g_2, & x \in \partial\Omega, \end{cases}$$

with $g_1, g_2 \in C^0(\partial\Omega)$ and $\|g_1 - g_2\|_{C^0(\partial\Omega)} \leqslant \varepsilon$, then

$$\|u_{g_1} - u_{g_2}\|_{C^0(\bar{\Omega})} \leqslant \|g_1 - g_2\|_{C^0(\partial\Omega)} \leqslant \varepsilon,$$

i.e., small perturbations on the data imply small perturbations on the solution. Equivalently, the solution depends continuously on the data (the map $g \mapsto u_g$ is continuous in the $C^0$ topology).

2.3.5. *Uniqueness via energy methods.* We mention also an alternative way, based on the so called *energy estimates*, to establish uniqueness of solutions for either the Dirichlet, the Neumann, or the Robin problems.

Energy estimates are obtained by multiplying the equation $-\Delta u = 0$ by $u$ on both sides, integrating over the domain and integrating by parts. If $u_1, u_2 \in C^2(\bar{\Omega})$ are two classical solutions of the boundary value problem and we apply the procedure to their difference, $w = u_1 - u_2$, which also satisfies the equation $-\Delta w = 0$, we obtain

$$0 = \int_\Omega w\Delta w \,\mathrm{d}x = -\int_\Omega \nabla w \cdot \nabla w \,\mathrm{d}x + \int_{\partial\Omega} w\partial_\nu w \,\mathrm{d}S \tag{2.5}$$

(1) **Dirichlet problem:** Since $w = 0$ on $\partial\Omega$, from (2.5) yields

$$\int_\Omega |\nabla w|^2 = \int_{\partial\Omega} w\partial_\nu w = 0$$

Since $\Omega$ is connected, this implies that $w$ is constant in $\Omega$ and being $w = 0$ on $\partial\Omega$ we conclude $w = u_1 - u_2 = 0$ in $\bar{\Omega}$.

(2) **Neumann problem:** Since $\partial w = 0$ on $\partial\Omega$,

$$\int_\Omega |\nabla w|^2 = \int_{\partial\Omega} w\partial_\nu w = 0.$$

hence, again, $w$ is constant in $\Omega$. However, this time the value of the solution on the boundary is not fixed, so the only conclusion that we can draw is that any two solutions $u_1$, $u_2$ differ for a constant value or, equivalently, the solution to the Neumann problem is unique up to an additive constant. We note, on the other hand, that the boundary datum $h$ cannot be chosen arbitrarily for the solution to exist and has to satisfy a compatibility condition

$$\int_{\partial\Omega} h = \int_{\partial\Omega} \partial_\nu u = \int_\Omega \Delta u = 0.$$

(3) **Robin problem:** Since $\partial_\nu w + \alpha\, w = 0$ on $\partial\Omega$,

$$\int_\Omega |\nabla w|^2 + \alpha \int_{\partial\Omega} w^2 = 0,$$

which yields $|\nabla w| = 0$ in $\Omega$ and $w = 0$ on $\partial\Omega$, hence again $w = u_1 - u_2 = 0$ in $\bar{\Omega}$ and uniqueness of solutions.

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2.3.6. *Alternative proof of the strong maximum principle.* Next we present another proof, based on a key lemma due to[10] Hopf[11] [Hop27] and Oleĭnik[12] [Ole52] (see also [AN16; Naz12; PS04b; PS04a] for further discussion) which does not rely on the mean-value property.

Lemma 2.1 (Zaremba–Hopf–Oleĭnik's boundary point lemma). *Let us suppose that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is sub-harmonic in an open set $\Omega$ and $u(x) < M$ for every $x \in \Omega$. If $u(\bar{x}) = M$ for some $\bar{x} \in \partial\Omega$ and $\Omega$ satisfies the interior sphere condition*[13] *at $\bar{x}$, then $\partial_\nu u(\bar{x}) > 0$, where $\partial_\nu$ is the derivative in the outward unit normal direction to a sphere that touches $\partial\Omega$ at $\bar{x}$.*

Proof. By the interior sphere condition, there is a ball $B_R(x) \subset \Omega$ with $\bar{x} \in \partial B_R(x)$. Let $M' := \max_{\overline{B}_{R/2}(x)} u < M$ and define $\varepsilon := M - M' > 0$. We consider a perturbation

$$w = u + \varepsilon v - M,$$

of $u$, where $v \in C^2(\mathbb{R}^n)$ to have the following properties:

(1) $v = 0$ on $\partial B_R(x)$;
(2) $v = 1$ on $\partial B_{R/2}(x)$;
(3) $\partial_\nu v < 0$ on $\partial B_R(x)$;
(4) $-\Delta v \leqslant 0$ in $B_R(x) \backslash \overline{B}_{R/2}(x)$.

Then $w \leqslant 0$ on $\partial B_R(x)$ and $\partial B_{R/2}(x)$, and $-\Delta w \leqslant 0$ in $B_R(x) \backslash \overline{B}_{R/2}(x)$. The weak maximum principle for sub-harmonic functions in Theorem 2.11 implies that $w \leqslant 0$ in $B_R(x) \backslash \overline{B}_{R/2}(x)$. Since $w(\bar{x}) = 0$, it follows that $\partial_\nu w(\bar{x}) \geqslant 0$. Therefore,

$$\partial_\nu u(\bar{x}) = \partial_\nu w(\bar{x}) - \varepsilon \partial_\nu v(\bar{x}) > 0,$$

which proves the result.

Let us give an explicit example of the perturbation $v$ (considering $B_R(0)$, without loss of generality):

$$v(x) = c\left[e^{-\alpha|x|^2} - e^{-\alpha R^2}\right],$$

where $c, \alpha$ are suitable positive constants. We have $v(x) = 0$ on $|x| = R$, and by choosing

$$c = \frac{1}{e^{-\alpha R^2/4} - e^{-\alpha R^2}},$$

we ensure that $v(R/2) = 1$. We compute

$$\partial_\nu v(x) = -2c\alpha|x|e^{-\alpha|x|^2} < 0 \quad \text{on } |x| = R.$$

and

$$\Delta v(x) = 2c\alpha\left[2\alpha|x|^2 - n\right]e^{-\alpha|x|^2}.$$

Thus, by choosing $\alpha \geqslant \frac{2n}{R^2}$, we obtain $-\Delta v < 0$ for $R/2 < |x| < R$.

□

Alternative proof of Theorem 2.10. As before, let

$$M := \max_{\bar{\Omega}} u$$

and define

$$F := \{x \in \Omega : u(x) = M\}.$$

Then $F$ is non-empty by assumption, and it is relatively closed in $\Omega$ since $u$ is continuous.

---

[10] A particular case, for the Laplace equation in a 3-dimensional domain, of this lemma is due to Stanisław Zaremba [Zar10].

[11] Eberhard Hopf (known for the Hopf maximum principle, the Hopf bifurcation theorem, the Wiener–Hopf method in integral equations, and the Cole–Hopf transformation for solving the viscous Burgers equation), not to be confused with Heinz Hopf (known for the Hopf–Rinow theorem, the Hopf fibration, and Hopf algebras.

[12] Olga Oleĭnik.

[13] We say that an open set $\Omega$ satisfies the *interior sphere condition* at $\bar{x} \in \partial\Omega$ if there is an open ball $B_r(x)$ contained in $\Omega$ such that $\bar{x} \in \partial B_r(x)$. Note that the interior sphere condition is satisfied by open sets with a $C^2$-boundary, but it need not be satisfied by open sets with a $C^1$-boundary, and in that case the conclusion of the Hopf lemma may not hold.

Let us suppose, for the sake of finding a contradiction, that $F \neq \Omega$. Then

$$G = \Omega \backslash F$$

is non-empty and open, and the boundary $\partial F \cap \Omega = \partial G \cap \Omega$ is non-empty (otherwise $F$ and $G$ are open and $\Omega$ is not connected).

Choose $y \in \partial G \cap \Omega$ and let $d = \operatorname{dist}(y, \partial\Omega) > 0$. Then choose $x \in G$ such that $|x - y| < \frac{d}{2}$ and let $r = \operatorname{dist}(x, F)$, which satisfies $0 < r < \frac{d}{2}$, so $\overline{B}_r(x) \subset G$. Moreover, there exists at least one point $\bar{x} \in \partial B_r(x) \cap \partial G$ such that $u(\bar{x}) = M$.

We therefore have the following situation: $u$ is sub-harmonic in an open set $G$ where $u < M$, the ball $B_r(x)$ is contained in $G$, and $u(\bar{x}) = M$ for some point $\bar{x} \in \partial B_r(x) \cap \partial G$. Lemma 2.1 then implies that

$$\partial_\nu u(\bar{x}) > 0$$

where $\partial_\nu$ is the outward unit normal derivative to the sphere $\partial B_r(x)$.

However, since $\bar{x}$ is an interior point of $\Omega$ and $u$ attains its maximum value $M$ there, we have $\nabla u(\bar{x}) = 0$, so $\partial_\nu u(\bar{x}) = \nabla u(\bar{x}) \cdot \nu = 0$, which is a contradiction.

If $\Omega$ is not connected, then $u$ is constant in any connected component of $\Omega$ that contains an interior point where $u$ attains a maximum value. □

**2.4. Harnack's inequality.** The maximum principle gives a basic pointwise estimate for solutions of Laplace's equation. Harnack's inequality[14] is another useful pointwise estimate. It states that if a function is nonnegative and harmonic in a domain, then the ratio of the maximum and minimum of the function on a compactly supported subdomain is bounded by a constant that depends only on the domains. This inequality controls, for example, the amount by which a harmonic function can oscillate inside a domain in terms of the size of the function.

THEOREM 2.12 (Harnack's inequality). *Suppose that $\Omega' \subset\subset \Omega$ is a connected open set that is compactly contained in an open set $\Omega$. There exists a constant $C$, depending only on $\Omega$ and $\Omega'$, such that if $u \in C(\Omega)$ is a non-negative function with the mean-value property (i.e., a non-negative harmonic function), then*

$$\sup_{\Omega'} u \leqslant C \inf_{\Omega'} u. \tag{2.6}$$

PROOF. Step 1. First, we establish the inequality for a compactly contained open ball. Suppose that $x \in \Omega$ and $B_{4R}(x) \subset\subset \Omega$, and let $u$ be any non-negative function with the mean-value property in $\Omega$. If $y \in B_R(x)$, then

$$u(y) = \fint_{B_R(y)} u \, dx \leqslant 2^n \fint_{B_{2R}(x)} u \, dx,$$

since $B_R(y) \subset B_{2R}(x)$ and $u$ is non-negative. Similarly, if $z \in B_R(x)$, then

$$u(z) = \fint_{B_{3R}(z)} u \, dx \geqslant \left(\frac{2}{3}\right)^n \fint_{B_{2R}(x)} u \, dx,$$

since $B_{3R}(z) \supset B_{2R}(x)$. It follows that

$$\sup_{B_R(x)} u \leqslant 3^n \inf_{B_R(x)} u.$$

Step 2. Suppose that $\Omega' \subset\subset \Omega$ and $0 < 4R < \operatorname{dist}(\Omega', \partial\Omega)$. Since $\bar{\Omega}'$ is compact, we may cover $\bar{\Omega}'$ by a finite number of open balls of radius $R$, where the number $N$ of such balls depends only on $\Omega'$ and $\Omega$. Moreover, since $\Omega'$ is connected, for any $x, y \in \Omega$ there is a sequence of at most $N$ overlapping balls $\{B_1, B_2, \dots, B_k\}$ such that $B_i \cap B_{i+1} \neq \varnothing$ and $x \in B_1$, $y \in B_k$. Applying the above estimate to each ball and combining the results, we obtain that

$$\sup_{\Omega'} u \leqslant 3^{nN} \inf_{\Omega'} u.$$

□

[14] Named after Carl Gustav Axel Harnack.

Harnack's inequality leads to an important convergence theorem for harmonic functions known as Harnack's principle or Harnack's convergence theorem. Consider a monontone sequence of continuous functions on $\Omega$. The pointwise limit of such a sequence need not behave well—it could be infinite at some points and finite at other points. Even if it is finite everywhere, there is no reason to expect that our sequence converges uniformly on every compact subset of $\Omega$. Harnack's principle shows that this kind of pathologies cannot occur for a monotone sequence of harmonic functions.

THEOREM 2.13 (Harnack's convergence theorem). *Suppose $\Omega$ is connected and $\{u_m\}_{m\in\mathbb{N}}$ is a pointwise increasing sequence of harmonic functions on $\Omega$. Then either $\{u_m\}_{m\in\mathbb{N}}$ converges uniformly on compact subsets of $\Omega$ to a function harmonic on $\Omega$, or $u_m(x) \to \infty$ for every $x \in \Omega$.*

PROOF. Replacing $u_m$ by $u_m - u_1 + 1$, we can assume that each $u_m$ is positive on $\Omega$ (since $\{u_m\}_{m\in\mathbb{N}}$ is monotone increasing). Set $u(x) = \lim_{m\to\infty} u_m(x)$ for each $x \in \Omega$.

Case 1. First suppose $u$ is finite everywhere on $\Omega$. Let $K$ be a compact subset of $\Omega$. Fix $x \in K$. Harnack's inequality shows there is a constant $C \in (1, \infty)$ such that

$$u_m(y) - u_k(y) \leqslant C(u_m(x) - u_k(x))$$

for all $y \in K$, whenever $m > k$. This implies $(u_m)$ is uniformly Cauchy on $K$, and thus $u_m \to u$ uniformly on $K$, as desired. Thanks to Problem 1 in Exercise Sheet 2, we have that the limit function $u$ is harmonic on $\Omega$.

Case 2. Now suppose $u(x) = \infty$ for some $x \in \Omega$. Let $y \in \Omega$. Then Harnack's inequality, applied to the compact set $K = \{x, y\}$, shows that there is a constant $C \in (1, \infty)$ such that $u_m(x) \leqslant Cu_m(y)$ for every $m$. Because $u_m(x) \to \infty$, we also have $u_m(y) \to \infty$, and so $u(y) = \infty$. This implies that $u$ is identically $\infty$ on $\Omega$. □

## 3. An interlude about distributions

The theory of distributions[15] is a powerful theory that allows one to extend the definition of derivative of a function beyond the classical sense.

Let us consider a function $f \in C^1(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$, where $C_c^\infty(\Omega)$ denotes the space of $C^\infty$ functions compactly supported in $\Omega$. The integration by parts formula gives in this case

$$\int_\Omega \frac{\partial f}{\partial x_i}\varphi \,\mathrm{d}x = -\int_\Omega f \frac{\partial \varphi}{\partial x_i}\,\mathrm{d}x$$

where the boundary term disappears since $\varphi$ vanishes on $\partial\Omega$. If now $f \notin C^1(\Omega)$, the idea is to define $\frac{\partial f}{\partial x_i}$ via (2.21), i.e., we define $\frac{\partial f}{\partial x_i}$ as an object that, when integrated (tested) against smooth functions with compact support, returns

$$\left\langle \frac{\partial f}{\partial x_i}, \varphi \right\rangle := -\int_\Omega f \frac{\partial \varphi}{\partial x_i}\,\mathrm{d}x.$$

The symbol $\langle \cdot, \cdot \rangle$ is used here to denote the action of the derivative $\frac{\partial f}{\partial x_i}$ on the test function $\varphi$.

DEFINITION 2.4 (Space of test functions). *We denote by $\mathcal{D}(\Omega)$, called the space of test functions, the space $C_c^\infty(\Omega)$ of infinitely smooth functions with compact support in $\Omega$, endowed with the following notion of convergence: given $\{\varphi_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, we say that $\varphi_k \to \varphi$ in $\mathcal{D}(\Omega)$ as $k \to \infty$ if*

*(1) there exists a compact subset $K \subset \Omega$ such that $\operatorname{supp}(\varphi_k) \subset K$ for all $k \in \mathbb{N}$;*
*(2) $D^\alpha \varphi_k \to D^\alpha \varphi$ uniformly in $K$, for all $\alpha \in \mathbb{N}^n$.*

One can construct a topology $\mathcal{T}$ on $\mathcal{D}(\Omega)$ which is consistent with the notion of convergence given in Definition 2.4. The topological space $(\mathcal{D}(\Omega), \mathcal{T})$ is complete, however non-metrizable.

DEFINITION 2.5 (Distribution). *A distribution $T$ in $\Omega$ is a functional $T : \mathcal{D}(\Omega) \to \mathbb{R}$ that satisfies the following properties:*

- *linearity: $\langle T, \alpha\varphi + \beta\psi \rangle = \alpha\langle T, \varphi \rangle + \beta\langle T, \psi \rangle$, for all $\varphi, \psi \in \mathcal{D}(\Omega), \alpha, \beta \in \mathbb{R}$;*
- *sequential continuity: $\langle T, \varphi_k \rangle \to \langle T, \varphi \rangle$ whenever $\varphi_k \to \varphi$ in $\mathcal{D}(\Omega)$.*

*The set of all distributions is denoted by $\mathcal{D}'(\Omega)$.*

[15] Laurent Schwartz was awarded the Fields Medal for his work on distributions in 1950 [Sch50; Sch51; Sch66].

We say that two distributions $T_1, T_2 \in \mathcal{D}'(\Omega)$ are equal if

$$\langle T_1, \varphi \rangle = \langle T_2, \varphi \rangle, \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

One can show that linear functionals on $(\mathcal{D}(\Omega), \mathcal{T})$ are continuous if and only if they are sequentially continuous. The space $\mathcal{D}'(\Omega)$ thus coincides then with the topological dual of $\mathcal{D}(\Omega)$.

EXAMPLE 2.18. *As a first example of distribution, let us consider a locally integrable function $f \in L^1_{loc}(\Omega)$. To it, we can associate a distribution $T_f : \mathcal{D}(\Omega) \to \mathbb{R}$ defined as*

$$\langle T_f, \varphi \rangle = \int_\Omega f\varphi \,\mathrm{d}x, \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

*The distribution $T_f$ is clearly a linear functional. It is also sequentially continuous in $\mathcal{D}(\Omega)$. Indeed, let $\varphi_k \to \varphi$ in $\mathcal{D}(\Omega)$; in particular, there exists $K \subset\subset \Omega$ such that $\operatorname{supp} \varphi_k \subset K$, for all $k$. (Here $A \subset\subset B$ means that $A$ has compact closure in $B$) and $\varphi_k \to \varphi$ uniformly in $K$. It follows that*

$$|\langle T_f, \varphi_k \rangle - \langle T_f, \varphi \rangle| = \left| \int_\Omega f(\varphi_k - \varphi)\,\mathrm{d}x \right| \leqslant \|f\|_{L^1(K)} \|\varphi_k - \varphi\|_{L^\infty(K)} \to 0 \qquad \text{as } k \to \infty.$$

*With some abuse of notation, we will denote by $f$ both the function in $L^1_{loc}(\Omega)$ and the corresponding distribution $T_f \in \mathcal{D}'(\Omega)$.*

EXAMPLE 2.19. *As a second example of distribution, we define the Dirac mass or Dirac distribution at $y$, denoted $\delta_y$: it is the distribution that satisfies*

$$\langle \delta_y, \varphi \rangle = \varphi(y), \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

*Again, the functional $\delta_y : \mathcal{D}(\Omega) \to \mathbb{R}$ is clearly linear. It is also sequentially continuous in $\mathcal{D}(\Omega)$ since, for $\varphi_k \to \varphi$ in $\mathcal{D}(\Omega)$, it holds*

$$\langle \delta_y, \varphi_k \rangle = \varphi_k(y) \to \varphi(y) = \langle \delta_y, \varphi \rangle.$$

*We conclude then that $\delta_y \in \mathcal{D}'(\Omega)$.*

On the space of distributions, we can introduce the following notion of (weak) convergence.

DEFINITION 2.6 (Convergence in distribution). *Let $\{T_k\}_{k \in \mathbb{N}} \subset \mathcal{D}'(\Omega)$ and $T \in \mathcal{D}'(\Omega)$. We say that $T_k \to T$ in $\mathcal{D}'(\Omega)$ as $k \to \infty$ if*

$$\langle T_k, \varphi \rangle \to \langle T, \varphi \rangle, \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

We now introduce the notion of *distributional derivative*.

DEFINITION 2.7 (Distributional derivatives). *Given a distribution $T$, for $\alpha \in \mathbb{N}^n$, the $\alpha$-*distributional derivative *$D^\alpha T \in \mathcal{D}'(\Omega)$ is defined as*

$$\langle D^\alpha T, \varphi \rangle := (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle, \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

EXAMPLE 2.20. *In the particular case $f \in L^1_{loc}(\Omega)$, the distributional partial derivative $\partial_{x_i} f \in \mathcal{D}'(\Omega)$ is defined as*

$$\langle \partial_{x_i} f, \varphi \rangle := -\int_\Omega f \partial_{x_i} \varphi \,\mathrm{d}x, \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

**Lect. 5, 08.10**

PROPOSITION 2.1. *Let $(T_k)_{n \geqslant 0}$ be a sequence in $\mathcal{D}'(\Omega)$ that converges to $T \in \mathcal{D}'(\Omega)$. Then, for all $\alpha \in \mathbb{N}^n$, $(\partial^\alpha T_k)_{k \geqslant 0}$ converges to $\partial^\alpha T$ in $\mathcal{D}'(\Omega)$.*

PROOF. Let $\varphi \in C^\infty_c(\Omega)$. For all $k \in \mathbb{N}$,

$$\langle \partial^\alpha T_k, \varphi \rangle = (-1)^{|\alpha|} \langle T_k, \partial^\alpha \varphi \rangle \longrightarrow (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle = \langle \partial^\alpha T, \varphi \rangle \qquad \text{as } k \to \infty.$$

□

EXAMPLE 2.21. *The distributional derivative of the Heaviside function*[16]

$$H(x) = \begin{cases} 1 & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0, \end{cases}$$

[16] Named after Oliver Heaviside.

*is given by the Delta distribution* $\delta_0$. *Indeed,*

$$\langle H', \varphi\rangle = -\int_{\mathbb{R}} H(x)\varphi'(x)\,\mathrm{d}x = -\int_0^\infty \varphi'(x)\,\mathrm{d}x = \varphi(0), \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

EXAMPLE 2.22. *The function defined for* $x \neq 0$ *by* $f(x) = \log|x|$ *and assigned any arbitrary value at* 0 *belongs to* $L^1_{\mathrm{loc}}(\mathbb{R})$. *Therefore, it can be associated with a distribution* $T_f \in \mathcal{D}'(\mathbb{R})$. *Let us compute its distributional derivative. For* $\varphi \in C_c^\infty(\mathbb{R})$, *we have*

$$\langle f', \varphi\rangle = -\langle f, \varphi'\rangle = -\int_{\mathbb{R}} \log|x| \cdot \varphi'(x)\,\mathrm{d}x$$

*Due to the integrability of the logarithm at* 0, *we have*

$$\begin{aligned} -\int_{\mathbb{R}} \log|x| \cdot \varphi'(x)\,\mathrm{d}x &= -\lim_{\varepsilon\to 0}\int_{|x|\geqslant\varepsilon} \log|x|\cdot\varphi'(x)\,\mathrm{d}x \\ &= -\lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-\varepsilon}\log(-x)\cdot\varphi'(x)\,\mathrm{d}x + \int_\varepsilon^\infty \log(x)\cdot\varphi'(x)\,\mathrm{d}x\right) \end{aligned}$$

*Integrating by parts yields*

$$\begin{aligned} &\int_{|x|\geqslant\varepsilon} \log|x|\cdot\varphi'(x)\,\mathrm{d}x \\ &= -\int_{|x|\geqslant\varepsilon}\frac{\varphi(x)}{x}\,\mathrm{d}x + \varphi(-\varepsilon)\log(\varepsilon) - \varphi(\varepsilon)\log(\varepsilon) \\ &= -\int_{|x|\geqslant\varepsilon}\frac{\varphi(x)}{x}\,\mathrm{d}x - 2\varepsilon\log(\varepsilon)\left(\frac{\varphi(-\varepsilon)+\varphi(\varepsilon)}{2\varepsilon}\right). \end{aligned}$$

*Since* $\varepsilon\log(\varepsilon)\to 0$ *as* $\varepsilon \to 0$ *and* $\varphi'$ *is bounded, we conclude that*

$$\begin{aligned} \langle f', \varphi\rangle &= -\lim_{\varepsilon\to 0}\int_{|x|\geqslant\varepsilon}\log|x|\cdot\varphi'(x)\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0}\int_{|x|\geqslant\varepsilon}\frac{\varphi(x)}{x}\,\mathrm{d}x =: \left\langle \text{p.v.}\left(\frac{1}{x}\right), \varphi\right\rangle. \end{aligned}$$

*The expression* p.v. *stands for* principal value. *If* $f$ *has an isolated singularity at the origin but is* $C^\infty$ *away from it, then the principal-value distribution of* $f$ *is defined by*

$$\langle \text{p. v.}(f), \varphi\rangle = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(0)} f(x)\varphi(x)\,\mathrm{d}x, \qquad \text{for all } \varphi\in\mathcal{D}(\Omega).$$

*Such a limit may not be well-defined, or, being well-defined, it may not necessarily define a distribution. It is, however, well-defined if* $f$ *is a continuous homogeneous function of degree* $-n$.

EXAMPLE 2.23. *Let* $u \in L^1_{\mathrm{loc}}(\mathbb{R})$, *and define, for* $x \in \mathbb{R}$, $v(x) = \int_0^x u(t)\,\mathrm{d}t$. *Then* $v$ *is a continuous function on* $\mathbb{R}$ *and* $v' = u$ *in the sense of distributions.*

*Let us first show the continuity of* $v$. *Let* $x_0 \in \mathbb{R}$ *and let* $\{x_k\}_{n\geqslant 0}$ *be a sequence converging to* $x_0$. *We have, for all* $n \geqslant 0$,

$$v(x_k) = \int_{\mathbb{R}} \chi_{(0,x_k)}(t)u(t)\,\mathrm{d}t$$

*By Lebesgue's dominated convergence theorem, the sequence* $(v(x_k))_{n\geqslant 0}$ *converges to* $\int_{\mathbb{R}}\chi_{(0,x_0)}(t)u(t)\,\mathrm{d}t = v(x_0)$, *proving the continuity of* $v$ *at* $x_0$, *and thus on* $\mathbb{R}$.

*Let* $\varphi \in C_c^\infty(\mathbb{R})$ *and assume that* $\mathrm{supp}(\varphi) \subset [-A, A]$. *Using Fubini's theorem, we have:*

$$\begin{aligned} \langle v', \varphi\rangle = -\langle v, \varphi'\rangle &= -\int_{-A}^A\left(\int_0^x u(t)\,\mathrm{d}t\right)\varphi'(x)\,\mathrm{d}x \\ &= -\int_0^A\int_0^x u(t)\varphi'(x)\,\mathrm{d}t\,\mathrm{d}x + \int_{-A}^0\int_x^0 u(t)\varphi'(x)\,\mathrm{d}t\,\mathrm{d}x \\ &= -\int_0^A u(t)\left(\int_t^A\varphi'(x)\,\mathrm{d}x\right)\mathrm{d}t + \int_{-A}^0 u(t)\left(\int_{-A}^t\varphi'(x)\,\mathrm{d}x\right)\mathrm{d}t \\ &= \int_0^A u(t)\varphi(t)\,\mathrm{d}t + \int_{-A}^0 u(t)\varphi(t)\,\mathrm{d}t = \int_{\mathbb{R}} u(t)\varphi(t)\,\mathrm{d}t = \langle u, \varphi\rangle. \end{aligned}$$

*Thus,* $v' = u$ *in* $\mathcal{D}'(\mathbb{R})$.

PROPOSITION 2.2. *Let* $a \in C^\infty(\Omega)$ *and let* $T \in \mathcal{D}'(\Omega)$*. Then, defining the product* $aT$ *as*

$$\langle aT, \varphi \rangle := \langle T, a\varphi \rangle, \qquad \text{for all } \varphi \in \mathcal{D}(\Omega),$$

*we have* $\partial_{x_i}(aT) = (\partial_{x_i} a)T + a\partial_{x_i} T$.

EXAMPLE 2.24. *The* $i$*-th partial derivative of a distribution* $T_f$ *with* $f \in C^1(\mathbb{R}^n)$ *is the distribution* $T_{\partial_{x_i} f}$.

EXAMPLE 2.25. *Let* $f$ *be a piecewise* $C^1$ *function on* $[a, b]$*. This means that there exists a subdivision of* $[a, b]$ *into intervals* $[a_i, a_{i+1}]$ *such that* $f$ *is* $C^1$ *on* $[a_i, a_{i+1}]$*. Suppose that, at every point where it is not continuous,* $f$ *admits a right limit and a left limit. To fix notations, we let* $a_0 = a$, $a_{n+1} = b$*, and* $a_1, \dots, a_n \in (a, b)$*. We denote by* $f(a_i^+)$ *and* $f(a_i^-)$ *the right and left limits of* $f$ *at the point* $a_i$*, respectively. As a convention, we let* $f(a_0^-) = f(a_{n+1}^+) = 0$.

*Then, we claim that*

$$(T_f)' = T_{f'} + \sum_{i=1}^{n+1} (f(a_i^+) - f(a_i^-))\delta_{a_i}.$$

*First, we note that the function* $f$ *defines a distribution, of which we calculate the derivative, which we denote by* $(T_f)'$*. By definition, for* $\varphi \in C_c^\infty(\mathbb{R})$*, we have*

$$\langle (T_f)', \varphi \rangle = -\langle T_f, \varphi' \rangle = -\int_a^b f(x)\varphi'(x)\,\mathrm{d}x$$

*Thus,*

$$\int_a^b f(x)\varphi'(x)\,\mathrm{d}x = \sum_{i=0}^{n} \int_{a_i}^{a_{i+1}} f(x)\varphi'(x)\,\mathrm{d}x.$$

*Integrating by parts, we get*

$$\int_{a_i}^{a_{i+1}} f(x)\varphi'(x)\,\mathrm{d}x = [f(x)\varphi(x)]_{a_i}^{a_{i+1}} - \int_{a_i}^{a_{i+1}} f'(x)\varphi(x)\,\mathrm{d}x,$$

*which yields*

$$-\int_a^b f(x)\varphi'(x)\,\mathrm{d}x = \sum_{i=0}^{n} \left[ f(a_i^+)\varphi(a_i^+) - f(a_{i+1}^-)\varphi(a_{i+1}^-) \right] + \int_{a_i}^{a_{i+1}} f'(x)\varphi(x)\,\mathrm{d}x.$$

*Hence,*

$$\langle (T_f)', \varphi \rangle = \langle T_{f'}, \varphi \rangle + f(a_n^+)\varphi(a_n^+) - f(a_0^-)\varphi(a_0^-) + \sum_{i=1}^{n} (f(a_i^+) - f(a_i^-))\varphi(a_i),$$

*which implies our claim.*

REMARK 2.6. *To regularize a distribution, we can use the convolution with the standard mollifier: if* $T \in \mathcal{D}'(\Omega)$*, then*

$$\langle T * \eta_\varepsilon, \varphi \rangle = \langle T, \eta_\varepsilon(-\cdot) * \varphi \rangle, \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

*It can be shown that the convolution of a smooth, compactly supported function and a distribution is a smooth function, namely*

$$x \mapsto (T * \eta_\varepsilon)(x) = \langle T, \eta_\varepsilon(\cdot - x) \rangle.$$

*Furthermore, we note that*

$$T * \eta_\varepsilon \to T \qquad \text{in } \mathcal{D}'(\Omega) \text{ as } \varepsilon \to 0.$$

REMARK 2.7. *The standard mollifier* $\eta_\varepsilon$ *from Definition 2.2 converges in distribution to the Dirac delta as* $\varepsilon \to 0$*: for every* $\varphi \in \mathcal{D}(\Omega)$ *and* $y \in \Omega$*, we have*

$$\varphi_\varepsilon(y) = \int_\Omega \eta_\varepsilon(x - y)\varphi(x)\,\mathrm{d}x \to \varphi(y) \quad \text{as } \varepsilon \to 0;$$

*hence*

$$\langle \eta_\varepsilon(\cdot - y), \varphi \rangle = \varphi_\varepsilon(y) \to \varphi(y) = \langle \delta_y, \varphi \rangle \quad \text{as } \varepsilon \to 0.$$

*i.e.,* $\eta_\varepsilon(-y) \to \delta_y$ *in* $\mathcal{D}'(\Omega)$.

PROPOSITION 2.3. *Let $T \in \mathcal{D}'(\mathbb{R}^d)$, $\varphi \in C_c^\infty(\mathbb{R}^d)$, and $\alpha \in \mathbb{N}^d$. Then,*

$$\partial^\alpha (T * \varphi) = (\partial^\alpha T) * \varphi = T * (\partial^\alpha \varphi).$$

*More generally, for any decomposition of the multi-index $\alpha = \alpha_1 + \alpha_2$, we have*

$$\partial^\alpha (T * \varphi) = (\partial^{\alpha_1} T) * (\partial^{\alpha_2} \varphi).$$

PROPOSITION 2.4. *Let $T \in \mathcal{D}'(\mathbb{R})$. We have $T' = 0$ if and only if $T$ is constant.*

PROOF. If we assume that $T$ is constant, it is clear that $T' = 0$ since $\varphi$ has compact support. Conversely, suppose that $T' = 0$. Then, for all $\varphi \in C_c^\infty(\mathbb{R})$,

$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle = 0.$$

Thus, $T$ vanishes on all functions of the form $\varphi'$, where $\varphi \in C_c^\infty(\mathbb{R})$. Let us characterize these functions. We show that

$$\big(\psi = \varphi', \text{ with } \varphi \in C_c^\infty(\mathbb{R})\big) \iff \left(\psi \in C_c^\infty(\mathbb{R}) \text{ and } \int_{\mathbb{R}} \psi(x)\,\mathrm{d}x = 0\right). \tag{3.1}$$

The direct implication is clear because $\varphi$ has compact support. Conversely, we set $\varphi(x) = \int_{-\infty}^x \psi(t)\,\mathrm{d}t$ with $\operatorname{supp}\psi \in [-M, M]$. Then $\varphi \in C^\infty(\mathbb{R})$. If $x < -M$, then $\varphi(x) = 0$ (since $\psi$ vanishes on $]-\infty, x]$ in this case). If $x > M$, then (since $\int_x^{+\infty} \psi(t)\,\mathrm{d}t = 0$ and $\int_{\mathbb{R}} \psi(t)\,\mathrm{d}t = 0$ by assumption),

$$\varphi(x) = \int_{-\infty}^x \psi(t)\,\mathrm{d}t + 0 = \int_{-\infty}^x \psi(t)\,\mathrm{d}t + \int_x^{+\infty} \psi(t)\,\mathrm{d}t = \int_{\mathbb{R}} \psi(t)\,\mathrm{d}t = 0.$$

Thus, $\operatorname{supp}\varphi \subset [-M, M]$ and $\varphi \in C_c^\infty(\mathbb{R})$ and, of course, $\psi = \varphi'$.

We will use the equivalence (3.1) to conclude the proof. Let $\chi \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \chi(x)\,\mathrm{d}x = 1$. Let $\varphi \in C_c^\infty(\mathbb{R})$. Let us define

$$\psi(x) = \varphi(x) - \left(\int_{\mathbb{R}} \varphi(t)\,\mathrm{d}t\right)\chi(x), \qquad x \in \mathbb{R}.$$

Then $\psi \in C_c^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} \psi(t)\,\mathrm{d}t = 0$. Consequently, there exists $\varphi \in C_c^\infty(\mathbb{R})$ such that $\psi = \varphi'$ and $\langle T, \psi \rangle = 0$. Thus, by the linearity of $T$,

$$\langle T, \varphi \rangle = \langle T, \chi \rangle \cdot \int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = C \cdot \langle 1, \varphi \rangle = \langle C, \varphi \rangle$$

with $C := \langle T, \chi \rangle$ being a constant. Thus, we conclude that $T$ is constant. □

## 4. Distributional Laplacian and Weyl's lemma

Thanks to Definition 2.7, we can say that any distribution admits distributional derivatives of any order. In particular, the *distributional Laplacian* of $T \in \mathcal{D}'(\Omega)$ will be defined as

$$\langle \Delta T, \varphi \rangle = \langle T, \Delta\varphi \rangle, \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

We will say that a distribution $u \in \mathcal{D}'(\Omega)$ is a *distributional solution of the Laplace equation* if

$$\langle \Delta u, \varphi \rangle = 0 \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

We conclude by stating the *fundamental lemma of the calculus of variations*. This tool is typically used to transform the distributional formulation of a differential problem into the strong formulation using a priori knowledge on the regularity of the distributional solution.

LEMMA 2.2 (Fundamental lemma of the calculus of variations). *Let $\Omega \subset \mathbb{R}^n$, and suppose $f \in C(\Omega)$. Suppose also that*

$$\int_\Omega f\varphi\,\mathrm{d}x = 0$$

*for every $\varphi \in C_c^\infty(\Omega)$. Then $f = 0$ on $\Omega$.*

REMARK 2.8. *One could state a more general version of the fundamental lemma of the calculus of variations without requiring continuity, e.g., assuming $f \in L^1_{\mathrm{loc}}(\Omega)$ and reach the conclusion $f = 0$ almost everywhere. The argument is done by approximation, but we skip the details.*

PROOF. For the sake of finding a contradiction, suppose that $f$ is strictly positive at some point $\bar{x} \in \Omega$. By continuity, there exists some neighborhood $U = \|x - \bar{x}\| < \delta$ where $f$ is positive. Consider the function

$$\varphi(x) = \begin{cases} 0, & \text{if } x \notin U, \\ \prod_{j=1}^n e^{-1/(x_j - \bar{x}_j)^2}, & \text{if } x \in U, \end{cases}$$

which is continuous and infinitely differentiable on $\mathbb{R}^n$, and satisfies $\varphi(x) = 0$ for $x \in \partial\Omega$, so it satisfies the hypotheses required above. Furthermore,

$$\int_\Omega f\varphi \, \mathrm{d}x = \int_\Omega f\varphi \, \mathrm{d}x > 0,$$

which is a contradiction. □

EXAMPLE 2.26. *If $u \in C^\infty(\Omega)$ is a distributional solution of the Laplace equation, then, for every $\varphi \in \mathcal{D}(\Omega)$,*

$$\langle \Delta u, \varphi \rangle = \langle u, \Delta\varphi \rangle = \int_\Omega u\Delta\varphi \, \mathrm{d}x = \int_\Omega \Delta u \, \varphi \, \mathrm{d}x$$

*where we first used some abuse of notation to identify the distribution $u$ with the $L^1_{\mathrm{loc}}$ function $u$ and then we integrated by parts. By Lemma 2.2, we now conclude that $\Delta u = 0$ in $\Omega$.*

We can now strengthen the regularity result for harmonic functions shown in Section 2.1 and prove Weyl's lemma,[17] which states that every distributional solution of Laplace's equation is smooth.

THEOREM 2.14 (Weyl's lemma). *Let $u \in \mathcal{D}'(\Omega)$ and suppose that $\langle \Delta u, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\Omega)$. Then $u \in C^\infty(\Omega)$ and is harmonic in $\Omega$.*

PROOF. Let $\eta_\varepsilon$ be the standard mollifier and consider $u_\varepsilon := u * \eta_\varepsilon$. Then $u_\varepsilon \in C^\infty$ and $\Delta u_\varepsilon = (\Delta u) * \eta_\varepsilon = 0$. So $u_\varepsilon$, being smooth and harmonic, satisfies the mean-value property. Hence, arguing as in the proof of Theorem 2.4,

$$\begin{aligned} u_\varepsilon * \eta(x) &= \int_0^\infty r^{n-1}\eta(r) \, \mathrm{d}r \int_{\partial B_1(0)} u_\varepsilon(x - ry) \, \mathrm{d}S(y) \\ &= \omega_n u_\varepsilon(x) \int_0^\infty r^{n-1}\eta(r) \, \mathrm{d}r = u_\varepsilon(x) \int_{\mathbb{R}^n} \eta(y) \, \mathrm{d}y \\ &= u_\varepsilon(x) \end{aligned}$$

(recall that here $\eta$ denotes the standard mollified with $\varepsilon = 1$). Letting $\varepsilon \to 0$, we get $u = u * \eta \in C^\infty$. Moreover, by the fundamental lemma of the calculus of variations, $-\Delta u = 0$ (i.e., $u$ is harmonic in the classical sense). □

ALTERNATIVE PROOF OF THEOREM 2.14. Let $\{\eta_\varepsilon\}_{\varepsilon>0}$ be the standard mollifier. Recall that $\eta(x) = \eta(|x|) = \theta(|x|^2)$ Fix $\Omega' \subset \Omega$ and put $\varepsilon_0 = \mathrm{dist}(\Omega', \partial\Omega)$. For each $x \in \Omega'$ and $\varepsilon \in (0, \varepsilon_0)$ the function

$$y \mapsto \eta_\varepsilon(x - y)$$

belongs to $\mathcal{D}(\Omega)$ and so we may consider $\langle u, \eta_\varepsilon(x - \cdot) \rangle$.

We compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\varepsilon^{-n}\eta\left(\frac{x-y}{\varepsilon}\right)\right) &= -n\varepsilon^{-n-1}\eta\left(\frac{x-y}{\varepsilon}\right) - \varepsilon^{-n}\nabla\eta\left(\frac{x-y}{\varepsilon}\right) \cdot \frac{x-y}{\varepsilon} \\ &= -\frac{1}{\varepsilon^{n+1}} \operatorname{div}\left(\frac{x-y}{\varepsilon}\eta\left(\frac{x-y}{\varepsilon}\right)\right) \end{aligned}$$

Defining

$$\Theta(t) := \frac{1}{2}\int_t^\infty \theta(s) \, \mathrm{d}s,$$

we note that

$$-z\,\eta(z) = -z\,\theta(|z|^2) = \nabla\left(\Theta(|z|^2)\right),$$

[17] Named after Hermann Weyl [Wey40]. See also [Str08].

and then we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\varepsilon^{-n}\eta\left(\frac{x-y}{\varepsilon}\right)\right) = -\frac{1}{\varepsilon^{n+1}}\operatorname{div}\left(\frac{x-y}{\varepsilon}\,\eta\left(\frac{x-y}{\varepsilon}\right)\right) = \Delta_y\left(\varepsilon^{1-n}\Theta\left(\left(\frac{x-y}{\varepsilon}\right)^2\right)\right).$$

Since, by assumption, $u$ is harmonic in the sense of distributions, we deduce

$$\left\langle u, \Delta_y\left(\varepsilon^{1-n}\Theta\left(\left(\frac{x-y}{\varepsilon}\right)^2\right)\right)\right\rangle = 0.$$

Now, by considering difference quotients, we see that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\langle u, \eta_\varepsilon(x-\cdot)\rangle = \left\langle u, \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\eta_\varepsilon(x-\cdot)\right\rangle = \left\langle u, \Delta_y\left(\varepsilon^{1-n}\Theta\left(\left(\frac{x-y}{\varepsilon}\right)^2\right)\right)\right\rangle = 0.$$

Integrating between $\varepsilon$ and $\varepsilon_1$ (such that $\varepsilon$, $\varepsilon_1 \in (0,\varepsilon_0)$), we deduce

$$\langle u, \eta_\varepsilon(x-\cdot)\rangle = \langle u, \eta_{\varepsilon_1}(x-\cdot)\rangle$$

For $\varphi \in \mathcal{D}(\Omega')$, we have

$$\langle u * \eta_\varepsilon, \varphi\rangle = \int_{\Omega'}\langle u, \eta_\varepsilon(x-\cdot)\rangle\varphi(x)\,\mathrm{d}x = \int_{\Omega'}\langle u, \eta_{\varepsilon_1}(x-\cdot)\rangle\varphi(x)\,\mathrm{d}x.$$

Hence, as $\eta_\varepsilon * \varphi \to \varphi$ in $\mathcal{D}(\Omega)$ as $\varepsilon \to 0^+$, we get

$$\langle u, \varphi\rangle = \int_{\Omega'}\langle u, \eta_{\varepsilon_1}(x-\cdot)\rangle\varphi(x)\,\mathrm{d}x.$$

Consequently, $u|_{\Omega'} \in C^\infty(\Omega')$, and since $\Omega'$ was arbitrary, this concludes the proof. □

### 5. Fundamental solution of the Laplace equation

Motivated by the fact that harmonic functions are invariant under rotations (Problem 4 in Exercise Sheet 1), we want to construct a radially symmetric solution $u(x) = v(|x|)$, with $v : \mathbb{R}_+ \to \mathbb{R}$, to the Laplace equation in $\mathbb{R}^n$.

We denote by $r = |x| = \sqrt{\sum_{i=1}^n x_i^2}$ the radial coordinate and note that

$$\partial_{x_i} r = \frac{x_i}{r}, \qquad \partial_{x_i}^2 = \frac{1}{r} - \frac{x_i^2}{r^3}, \qquad i = 1,\dots,n.$$

Hence,

$$\partial_{x_i} u(x) = v'(r)\frac{x_i}{r}, \qquad \partial_{x_i}^2 u(x) = v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right).$$

Then the Laplace equation is reduced to

$$\Delta u(x) = \sum_{i=1}^n \partial_{x_i}^2 u = v''(r) + v'(r)\frac{n-1}{r} = 0. \tag{5.1}$$

If $v' \neq 0$, we can solve the ODE (5.1) by noticing that

$$\frac{v''}{v'} = \frac{1-n}{r},$$

which yields

$$\ln v' = (1-n)\ln r,$$

that is

$$v' = \frac{C}{r^{n-1}}.$$

Integrating once more, we conclude that (5.1) admits the non-constant solutions of the form

$$v(r) = \begin{cases} C_1 r + C_2 & \text{if } n = 1,\\ C_1 \log r + C_2, & \text{if } n = 2,\\ C_1 r^{2-n} + C_2, & \text{if } n \geqslant 3.\end{cases}$$

We define the *fundamental solution* or *free-space Green's function* $\Phi : \mathbb{R}^n \to \mathbb{R}$ of the Laplace equation[18] by

$$\Phi(x) := \begin{cases} -\frac{1}{2}|x|, & \text{if } n = 1, \\ -\frac{1}{2\pi}\log|x|, & \text{if } n = 2, \\ \frac{1}{n(n-2)\alpha_n|x|^{n-2}}, & \text{if } n \geqslant 3. \end{cases} \tag{5.2}$$

The fundamental solution satisfies

$$-\Delta\Phi = \delta_0$$

in the sense of distributions in $\mathbb{R}^n$, as we will show in Theorem 2.15.

**Lect. 6, 15.10**

REMARK 2.9 (Some properties of the fundamental solution). *We collect here some properties of the fundamental solution $\Phi$ of the Laplace equation.*

*(1) $\Phi$ is smooth away from the origin: $\Phi \in C^\infty(\mathbb{R}^n\backslash\{0\})$. In particular, for $x \neq 0$, we compute that*

$$\partial_{x_i}\Phi(x) = -\frac{1}{n\alpha_n}\frac{x_i}{|x|^n}, \tag{5.3}$$

$$\partial^2_{x_i x_j}\Phi(x) = \frac{x_i x_j}{\alpha_n|x|^{n+2}} - \frac{\delta_{ij}}{n\alpha_n|x|^n} \tag{5.4}$$

*(here*

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

*is the Kronecker[19] delta).*

*(2) From (5.4), we deduce that*

$$\Delta\Phi = 0 \quad \text{if } x \neq 0,$$

*so $\Phi$ is harmonic in any open set that does not contain the origin.*

*(3) The function $\Phi$ is homogeneous of degree $-n+2$, its first derivative is homogeneous of degree $-n+1$, and its second derivative is homogeneous of degree $n$.*

*(4) $\Phi$ is unbounded as $x \to 0$ with $\Phi(x) \to \infty$ as $|x| \to 0$. Nevertheless, $\Phi$ and $\nabla\Phi$ are locally integrable. For example, the local integrability of $\partial_i\Phi$ follows from (5.3) by noticing that*

$$|\partial_{x_i}\Phi(x)| \leqslant \frac{C_n}{|x|^{n-1}},$$

*holds and that $|x|^{n-1}$ is locally integrable on $\mathbb{R}^n$.*

*(5) The second partial derivatives of $\Phi$ are not locally integrable, however, since they are of the order $|x|^{-n}$ as $x \to 0$.*

*(6) For $x \neq 0$,*

$$\nabla\Phi \cdot \frac{x}{|x|} = -\frac{1}{n\alpha_n}\frac{1}{|x|^{n-1}}. \tag{5.5}$$

*Thus we get the following surface integral over a sphere centered at the origin with normal $\nu = \frac{x}{|x|}$:*

$$-\int_{\partial B_r(0)} \nabla\Phi \cdot \nu \,\mathrm{d}S = 1. \tag{5.6}$$

---

[18] In general, the fundamental solution $\Phi$ of a linear partial differential operator $\mathcal{L}$ is the distributional solution of $\mathcal{L}\Phi = \delta_0$. The existence of fundamental solutions for every linear partial differential equation with constant coefficients was first proved independently by Bernard Malgrange [Mal56, Th. 1] and Leon Ehrenpreis [Ehr54, Th. 6]. Before 1950, when the first edition of [Sch50] appeared, the question about the existence of a fundamental solution was not even raised, since there did not exist a generally accepted definition of a fundamental solution. By now there are several proofs of Malgrange–Ehrenpreis' theorem, which can be roughly classified into three categories: (1) non-constructive proofs using Hahn–Banach's theorem; (2) constructive proofs via explicit formulae; (3) proofs by solving a "division problem". See [OW96; Ros91; Wag09] for further information.

[19] Named after Leopold Kronecker.

(7) *As follows from the divergence theorem and the fact that $\Phi$ is harmonic in $B_R(0)\backslash B_r(0)$, the integral (5.6) does not depend on $r$. The surface integral is not zero, however, as it would be for a function that was harmonic everywhere inside $B_r(0)$, including at the origin.*
(8) *The normalization of the flux integral in (5.6) to one accounts for the choice of the multiplicative constant in the definition of $\Phi$.*

THEOREM 2.15. *We have*

$$-\Delta\Phi = \delta_0$$

*in the sense of distributions, i.e.,*

$$-\langle\Phi, \Delta\varphi\rangle = \langle\delta_0, \varphi\rangle$$

*for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.*

We will prove Theorem 2.15 for $n \geqslant 2$. The case $n = 1$ is left as an exercise. Indeed, it follows directly from Problem 4 of Exercise Sheet 4: after computing the distributional derivatives $(T_{|\cdot|})' = T_{\text{sign}}$ and $(T_H)' = \delta_0$ (where $H = \chi_{[0,+\infty)}$ is the Heaviside function), one only has to notice the relationship between sign and Heaviside function, namely sign $= 2H - 1$, to conclude that

$$-\Delta\left(-\frac{1}{2}|x|\right) = \delta_0$$

in the sense of distributions.

PROOF. Let $T_\Phi$ be the distribution associated with the fundamental solution $\Phi$ (which is an $L^1_{\text{loc}}$-function). That is, let $T_\Phi : \mathcal{D}(\mathbb{R}^n) \to \mathbb{R}$ be defined as

$$\langle T_\Phi, g\rangle = \int_{\mathbb{R}^n} \Phi(x)g(x)\,\mathrm{d}x, \quad \text{for all } g \in \mathcal{D}(\mathbb{R}^n).$$

We want to show that

$$\langle T_\Phi, \Delta g\rangle = -\langle\delta_0, g\rangle = -g(0),$$

which means $-\Delta\Phi = \delta_0$ in $\mathbb{R}^n$ in the sense of distributions.

By definition,

$$\langle\Delta T_\Phi, g\rangle = \langle T_\Phi, \Delta g\rangle = \int_{\mathbb{R}^n} \Phi(x)\Delta g(x)\,\mathrm{d}x.$$

Now we would like to apply the divergence theorem, but $\Phi$ has a singularity at $x = 0$. We get around this by breaking up the integral into two pieces:

$$\begin{aligned}(T_\Phi, \Delta g) &= \int_{\mathbb{R}^n} \Phi(x)\Delta g(x)\,\mathrm{d}x \\ &= \int_{B_\delta(0)} \Phi(x)\Delta g(x)\,\mathrm{d}x + \int_{\mathbb{R}^n\backslash B_\delta(0)} \Phi(x)\Delta g(x)\,\mathrm{d}x \\ &=: I + J.\end{aligned}$$

Step 1. We look first at term $I$. For $n = 2$,

$$\begin{aligned}\left|-\int_{B_\delta(0)} \frac{1}{2\pi}\ln|x|\,\Delta g(x)\,\mathrm{d}x\right| &\leqslant C\,\|\Delta g\|_{L^\infty(\mathbb{R}^n)}\left|\int_{B_\delta(0)} \ln|x|\,\mathrm{d}x\right| \\ &\leqslant C\int_0^{2\pi}\int_0^\delta |\ln r|\,r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= C\int_0^\delta |\ln r|\,r\,\mathrm{d}r \\ &= C\delta^2|\ln\delta|.\end{aligned}$$

For $n \geqslant 3$,

$$\begin{aligned}
\left|\int_{B_\delta(0)} \frac{1}{n(n-2)\alpha_n} \frac{1}{|x|^{n-2}} \Delta g(x)\,\mathrm{d}x\right| &\leqslant C\,\|\Delta g\|_{L^\infty(\mathbb{R}^n)} \int_{B_\delta(0)} \frac{1}{|x|^{n-2}}\,\mathrm{d}x \\
&\leqslant C \int_0^\delta \left(\int_{\partial B_r(0)} \frac{1}{|y|^{n-2}}\,\mathrm{d}S(y)\right) \mathrm{d}r \\
&= C \int_0^\delta \frac{1}{r^{n-2}} \left(\int_{\partial B_r(0)} \mathrm{d}S(y)\right) \mathrm{d}r \\
&= C \int_0^\delta \frac{1}{r^{n-2}} n\alpha_n r^{n-1}\,\mathrm{d}r \\
&= n\alpha_n C \int_0^\delta r\,\mathrm{d}r \\
&= \frac{n\alpha_n C}{2}\delta^2.
\end{aligned}$$

Therefore, as $\delta \to 0^+$, $|I| \to 0$.

Step 2. Next, we look at term $J$. Applying the divergence theorem, we have

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus B_\delta(0)} \Phi(x)\Delta_x g(x)\,\mathrm{d}x &= \underbrace{\int_{\mathbb{R}^n \setminus B_\delta(0)} \Delta_x \Phi(x) g(x)\,\mathrm{d}x}_{=0} \\
&\quad - \int_{\partial(\mathbb{R}^n \setminus B_\delta(0))} \partial_\nu \Phi(x) g(x)\,\mathrm{d}S(x) + \int_{\partial(\mathbb{R}^n \setminus B_\delta(0))} \Phi(x)\partial_\nu g(x)\,\mathrm{d}S(x) \\
&= - \int_{\partial(\mathbb{R}^n \setminus B_\delta(0))} \partial_\nu \Phi(x) g(x)\,\mathrm{d}S(x) + \int_{\partial(\mathbb{R}^n \setminus B_\delta(0))} \Phi(x)\partial_\nu g\,\mathrm{d}S(x) \\
&=: J_1 + J_2,
\end{aligned}$$

using the fact that $\Delta_x \Phi(x) = 0$ for $x \in \mathbb{R}^n \setminus B_\delta(0)$.

Step 2a. We first look at term $J_1$. Since $g$ vanishes as $|x| \to \infty$, we only need to calculate the integral over $\partial B_\delta(0)$ where the normal vector $\nu$ is the outer normal to $\mathbb{R}^n \setminus B_\delta(0)$. From (5.3), we have

$$\nabla_x \Phi(x) = -\frac{x}{n\alpha_n |x|^n}.$$

The outer unit normal to $\mathbb{R}^n \setminus B_\delta(0)$ on $B_\delta(0)$ is given by $\nu = -\frac{x}{|x|}$, so the normal derivative of $\Phi$ on $B_\delta(0)$ is given by

$$\partial_\nu \Phi = \left(-\frac{x}{n\alpha_n |x|^n}\right) \cdot \left(-\frac{x}{|x|}\right) = \frac{1}{n\alpha_n |x|^{n-1}}$$

(as in (5.5)). Therefore, $J_1$ can be written as

$$-\int_{\partial B_\delta(0)} \frac{1}{n\alpha_n |x|^{n-1}} g(x)\,\mathrm{d}S(x) = -\frac{1}{n\alpha_n \delta^{n-1}} \int_{\partial B_\delta(0)} g(x)\,\mathrm{d}S(x).$$

Since $g$ is a continuous function, then we conclude

$$-\fint_{\partial B_\delta(0)} g(x)\,\mathrm{d}S(x) \to -g(0) \quad \text{as} \quad \delta \searrow 0.$$

Step 2b. Lastly, we look at term $J_2$. Using the fact that $g$ vanishes as $|x| \to \infty$, we only need to integrate over $\partial B_\delta(0)$. Since $g \in \mathcal{D}(\mathbb{R}^n)$, we compute

$$\left|\int_{\partial B_\delta(0)} \Phi(x)\partial_\nu g(x)\,\mathrm{d}S(x)\right| \leqslant \|\partial_\nu g\|_{L^\infty(\partial B_\delta(0))} \int_{\partial B_\delta(0)} |\Phi(x)|\,\mathrm{d}S(x) \leqslant C \int_{\partial B_\delta(0)} |\Phi(x)|\,\mathrm{d}S(x).$$

For $n = 2$,

$$
\begin{aligned}
\int_{\partial B_\delta(0)} |\Phi(x)| \, \mathrm{d}S(x) &= C \int_{\partial B_\delta(0)} |\ln |x|| \, \mathrm{d}S(x) \\
&\leqslant C|\ln \delta| \int_{\partial B_\delta(0)} \mathrm{d}S(x) \\
&= C|\ln \delta|(2\pi\delta) \leqslant C\delta|\ln \delta|.
\end{aligned}
$$

For $n \geqslant 3$,

$$
\begin{aligned}
\int_{\partial B_\delta(0)} |\Phi(x)| \, \mathrm{d}S(x) &= C \int_{\partial B_\delta(0)} \frac{1}{|x|^{n-2}} \, \mathrm{d}S(x) \\
&\leqslant \frac{C}{\delta^{n-2}} \int_{\partial B_\delta(0)} \mathrm{d}S(x) \\
&= \frac{C}{\delta^{n-2}} n\alpha_n \delta^{n-1} \leqslant C\delta.
\end{aligned}
$$

In both cases, we have $|J_2| \to 0$ as $\delta \searrow 0$.

Step 3. Combining these estimates, we see that

$$
\int_{\mathbb{R}^n} \Phi(x)\Delta_x g(x) \, \mathrm{d}x = \lim_{\delta \to 0^+} (I + J_1 + J_2) = -g(0),
$$

which concludes the proof. □

**5.1. Solution of the Poisson equation.** We can use the fundamental solution $\Phi$ to build a solution to the Poisson equation.

THEOREM 2.16. *Suppose that $f \in C_c^\infty(\mathbb{R}^n)$, and let*

$$
u = \Phi * f
$$

*where $\Phi$ is the fundamental solution (5.2). Then $u \in C^\infty(\mathbb{R}^n)$ and*

$$
-\Delta u = f \qquad \text{in } \mathbb{R}^n.
$$

PROOF. By a change of variables, we write

$$
u(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y) \, \mathrm{d}y = \int_{\mathbb{R}^n} \Phi(y) f(x - y) \, \mathrm{d}y.
$$

Let $e_i := (\ldots, 0, 1, 0, \ldots)$ be the unit vector in $\mathbb{R}^n$ with a 1 in the $i^{\text{th}}$ slot. Then

$$
\frac{u(x + he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y) \left[ \frac{f(x + he_i - y) - f(x - y)}{h} \right] \mathrm{d}y.
$$

Now $f \in C^2$ implies

$$
\frac{f(x + he_i - y) - f(x - y)}{h} \to \partial_{x_i} f(x - y), \quad \text{as } h \to 0, \text{ uniformly in } \mathbb{R}^n.
$$

Therefore,

$$
\partial_{x_i} u(x) = \int_{\mathbb{R}^n} \Phi(y) \partial_{x_i} f(x - y) \, \mathrm{d}y.
$$

Similarly, we also obtain

$$
\partial^2_{x_i x_j} u(x) = \int_{\mathbb{R}^n} \Phi(y) \partial^2_{x_i x_j} f(x - y) \, \mathrm{d}y.
$$

In particular,

$$
\Delta_x u(x) = \int_{\mathbb{R}^n} \Phi(y) \Delta_x f(x - y) \, \mathrm{d}y = \int_{\mathbb{R}^n} \Phi(y) \Delta_y f(x - y) \, \mathrm{d}y.
$$

Since $f \in \mathcal{D}(\mathbb{R}^n)$, we apply Theorem 2.15 and conclude that

$$
-\Delta u = \langle \delta_x, f \rangle = f(x).
$$

□

CHAPTER 3

# Classical solutions to the Dirichlet problem

## 1. Dirichlet problem for the Laplace equation

The fundamental solution $\Phi$ of the Laplace equation studied in Section 5 can be used to derive a representation formula for the point value of a $C^2(\Omega)$ function in terms of its Laplacian and boundary values.

THEOREM 3.1 (Green's representation formula). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^1$ boundary. For any $u \in C^2(\bar{\Omega})$ and $y \in \Omega$, we have*

$$\begin{aligned} u(y) = &\int_\Omega \Phi(x-y)(-\Delta u(x))\,\mathrm{d}x \\ &- \int_{\partial\Omega} \partial_{\nu_x}\Phi(x-y)u(x)\,\mathrm{d}S(x) + \int_{\partial\Omega} \Phi(x-y)\partial_\nu u(x)\,\mathrm{d}S(x), \end{aligned} \tag{1.1}$$

*where $\partial_{\nu_x}$ denotes the normal derivative with respect to the $x$ variables, i.e. $\partial_{\nu_x}(\cdot) = \nabla_x(\cdot)\cdot\nu = \sum_{i=1}^n \frac{\partial}{\partial x_i}(\cdot)\nu_i$.*

The starting point of the proof is Green's integration by parts identity,

$$\int_\Omega (v\Delta u - u\Delta v)\,\mathrm{d}x = \int_{\partial\Omega}(v\partial_\nu u - u\partial_\nu v)\,\mathrm{d}S. \tag{1.2}$$

which holds for every $u, v \in C^2(\bar{\Omega})$ and a bounded domain with $C^1$ boundary. Now fix $y \in \Omega$. To prove the (1.1), the idea is to take $v = \Phi(\cdot - y)$ in (1.2). However, $x \mapsto \Phi(x-y)$ is not $C^2(\bar{\Omega})$ and, in particular, a singularity occurs at $x = y$. So we need to be more careful: to circumvent this difficulty, we write the identity on the domain $\Omega_\varepsilon = \Omega\backslash B_\varepsilon(y)$, with $\varepsilon$ small enough so that $B_\varepsilon(y) \subset \Omega$, and let $\varepsilon \to 0$.

PROOF. Exploiting the fact that $\Phi(x-y) \in C^2(\bar{\Omega}_\varepsilon)$ and $\Delta_x\Phi(x-y) = 0$ in $\Omega_\varepsilon$, we have

$$\begin{aligned} &\underbrace{\int_{\Omega_\varepsilon} \Delta u(x)\Phi(x-y)\,\mathrm{d}x}_{=:C} - \int_{\partial\Omega}\left(\partial_\nu u(x)\Phi(x-y) - u(x)\partial_\nu\Phi(x-y)\right)\mathrm{d}S(x) \\ &= \underbrace{\int_{\partial B_\varepsilon(y)} \partial_\nu u(x)\Phi(x-y)\,\mathrm{d}S(x)}_{=:A} - \underbrace{\int_{\partial B_\varepsilon(y)} u(x)\partial_\nu\Phi(x-y)\,\mathrm{d}S(x)}_{=:B}. \end{aligned}$$

We show now that $A \to 0$, $B \to u(y)$, and $C \to \int_\Omega \Delta u(x)\Phi(x-y)\,\mathrm{d}x$ as $\varepsilon \to 0$.

For $A$, we estimate

$$|A| = |\Phi(\varepsilon)|\left|\int_{\partial B_\varepsilon(y)} \partial_\nu u\,\mathrm{d}S\right| \leqslant |\Phi(\varepsilon)|\varepsilon^{n-1}\omega_n \sup_{B_\varepsilon(y)}|\nabla u| \longrightarrow 0 \text{ as } \varepsilon \to 0.$$

For $B$, we estimate the difference $|B - u(y)|$:

$$
\begin{aligned}
|B - u(y)| &= \left| \int_{\partial B_\varepsilon(y)} \big(u(y) + \nabla u\big(y + \theta_x(x-y)\big) \cdot (x-y)\big)\, \partial_\nu \Phi(x-y)\, \mathrm{d}S(x) - u(y) \right| \\
&\leqslant \underbrace{\left| u(y) \left( \int_{\partial B_\varepsilon(y)} \partial_\nu \Phi(x-y)\, \mathrm{d}S(x) - 1 \right) \right|}_{=0} \\
&\quad + \left| \int_{\partial B_\varepsilon(y)} \nabla u(y + \theta_x(x-y)) \cdot (x-y) \partial_\nu \Phi(x-y)\, \mathrm{d}S(x) \right| \\
&\leqslant \sup_{B_\varepsilon(y)} |\nabla u| \varepsilon \int_{\partial B_\varepsilon(y)} |\partial_\nu \Phi(x-y)|\, \mathrm{d}S(x) \longrightarrow 0 \text{ as } \varepsilon \to 0..
\end{aligned}
$$

Notice that, in the second line, the fact that $\int_{\partial B_\varepsilon(y)} \partial_\nu \Phi(x-y)\, \mathrm{d}S(x) = 1$ is due to $\nu$ being the normal outgoing vector to the domain $\Omega \backslash B_\varepsilon(y)$, hence the normal *ingoing* vector to $B_\varepsilon(y)$.

Finally, for the term $C$, since $\Phi(x-y)$ is integrable in $\Omega$ for any $y \in \Omega$, we have

$$
\begin{aligned}
\left| C - \int_\Omega \Delta u(x) \Phi(x-y)\, \mathrm{d}x \right| &= \left| \int_{B_\varepsilon(y)} \Delta u(x) \Phi(x-y)\, \mathrm{d}x \right| \\
&\leqslant \sup_{B_\varepsilon(y)} |\Delta u| \int_{B_\varepsilon(y)} |\Phi(x-y)|\, \mathrm{d}x \\
&\leqslant \sup_{B_\varepsilon(y)} |\Delta u| \int_0^\varepsilon s^{n-1} \omega_n |\Phi(s)|\, \mathrm{d}s \longrightarrow 0 \qquad \text{as } \varepsilon \to 0.
\end{aligned}
$$

□

REMARK 3.1. *In these notes, we will use also the notation $\Phi(x, y) := \Phi(x-y) = \Phi(|x-y|)$, as needed.*

*Applying Green's representation formula to a test function $\varphi \in \mathcal{D}(\Omega)$, we obtain*

$$\varphi(y) = \int_\Omega \Phi(x, y) \Delta \varphi(x)\, \mathrm{d}x$$

*that is,*

$$\Delta_x \Phi(x, y) = \delta_y,$$

*which is consistent with Theorem 2.15.*

We may draw the following consequence from the Green representation formula: If one knows $\Delta u$, then $u$ is completely determined by its values and those of its normal derivative on $\partial\Omega$. In particular, a harmonic function on $\Omega$ can be reconstructed from its values on $\partial\Omega$. One may then ask conversely whether one can construct a harmonic function for arbitrary given values on $\partial\Omega$ for the function and its normal derivative.

DEFINITION 3.1 (Green function). *A function $G = G(x, y)$, defined for $x, y \in \bar{\Omega}$, with $x \neq y$, is called a Green function for $\Omega$ if*

*(1) $G(x, y) = 0$ for $x \in \partial\Omega$,*
*(2) $h(x, y) := G(x, y) - \Phi(x, y)$ is harmonic in $x \in \Omega$ (thus in particular also at the point $x = y$).*

Roughly speaking, (if it exists) $G(x, y) = \Phi(x, y) + h(x, y)$, where $h$ solves

$$
\begin{cases}
-\Delta_x h(x, y) = 0, & x \in \Omega, \\
h(x, y) = -\Phi(x, y), & x \in \partial\Omega.
\end{cases}
\tag{1.3}
$$

In other words, $G$ solves, for $y \in \Omega$,

$$
\begin{cases}
-\Delta_x G(x, y) = \delta_0(x-y), & x \in \Omega, \\
G(x, y) = 0, & x \in \partial\Omega.
\end{cases}
$$

We now assume that a Green function $G(x,y)$ for $\Omega$ exists and put $v(x) = h(x,y)$ in Green's identity (see Appendix 2) and subtract the result from (1.1), obtaining

$$u(y) = -\int_{\partial\Omega} u(x)\partial_{\nu_x} G(x,y)\,\mathrm{d}S(x) - \int_{\Omega} G(x,y)\Delta u(x)\,\mathrm{d}x. \tag{1.4}$$

We call $K(x,y) := -\partial_{\nu_x} G(x,y)$ the *Poisson kernel* of $\Omega$.

In particular, formula (1.4) implies that a harmonic $u$ is already determined by its boundary values $u_{|\partial\Omega}$. This construction now raises the converse question. If we are given functions $g : \partial\Omega \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$, can we obtain a solution of the Dirichlet problem for the Poisson equation

$$\begin{cases} -\Delta u(x) = f(x), & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega, \end{cases} \tag{1.5}$$

by the representation formula

$$u(y) = -\int_{\partial\Omega} \varphi(x)\partial_{\nu_x} G(x,y)\,\mathrm{d}S(x) + \int_{\Omega} f(x)G(x,y)\,\mathrm{d}x \tag{1.6}$$

derived above? One implication is already proven: if $u$ is a solution, it does satisfy this formula. For the other implication (namely, if $u$ satisfies the formula (1.6), then it is a solution of (1.5)), we will see that, essentially, the answer is yes, under suitable hypotheses.

We start our analysis by first considering the case $f \equiv 0$ and then extending it to a general $f$. In special domains, the Green function can be explicitly constructed.

**1.1. Dirichlet problem for the Laplace operator in a ball.** We start with the case $\Omega := B_R(0)$ and consider the problem

$$\begin{cases} -\Delta u(x) = 0, & x \in B_R(0), \\ u(x) = g(x), & x \in \partial B_R(0). \end{cases} \tag{1.7}$$

The idea is to construct $h$ by a proper reflection of $\Phi(\cdot - y)$.

For $y \in \mathbb{R}^n$, we put

$$\tilde{y} := \begin{cases} \frac{R^2}{|y|^2} y & \text{if } y \neq 0, \\ \infty & \text{if } y = 0, \end{cases}$$

($\tilde{y}$ is the point obtained from $y$ by reflection across $\partial B_R(0)$.) We then put

$$G(x,y) := \begin{cases} \Phi(|x-y|) - \Phi\left(\frac{|y|}{R}|x-\tilde{y}|\right), & \text{if } y \neq 0, \\ \Phi(|x|) - \Phi(R), & \text{if } y = 0. \end{cases} \tag{1.8}$$

For $x \neq y$, $G(x,y)$ is harmonic in $x$, since, for $y \in B_R(0)$, the point $\tilde{y}$ lies in the exterior of $B_R(0)$. The function $G(x,y)$ has only one singularity in $B_R(0)$, namely, at $x = y$, and this singularity is the same as that of $\Phi(x,y)$. The formula

$$G(x,y) = \Phi\left(\left(|x|^2 + |y|^2 - 2x\cdot y\right)^{1/2}\right) - \Phi\left(\left(\frac{|x|^2|y|^2}{R^2} + R^2 - 2x\cdot y\right)^{1/2}\right)$$

then shows that for $x \in \partial B_R(0)$, i.e., $|x| = R$, we have indeed $G(x,y) = 0$.

Therefore, the function $G$ defined by (1.8) is the Green function of $B_R(0)$. Equation (1.8) also implies the symmetry

$$G(x,y) = G(y,x).$$

We compute the partial derivatives of the Green function:

$$\partial_{x_i} G(x,y) = \frac{1}{\omega_n}\left(-\frac{x_i - y_i}{|x-y|^n} + \frac{R^{n-2}}{|y|^{n-2}}\frac{x_i - \tilde{y}_i}{|x-\tilde{y}|^n}\right);$$

since $\frac{|y|}{R}|x-\tilde{y}| = |x-y|$ whenever $x \in \partial B_R(0)$, we have, in particular,

$$\partial_{x_i} G(x,y) = -\frac{\left(1 - \frac{|y|^2}{R^2}\right)x_i}{\omega_n |x-y|^n}, \qquad x \in \partial B_R(0).$$

Finally, since $x \cdot \nu = R$, for $x \in \partial B_R(0)$, we have

$$\partial_{\nu_x} G(x,y) = -\frac{R^2 - |y|^2}{\omega_n R |x-y|^n}, \qquad x \in \partial B_R(0),\ y \in B_R(0).$$

In conclusion, we construct then a candidate solution to the Dirichlet problem (1.7) on the ball $B_R(0)$ with the formula

$$u(y) = \frac{R^2 - |y|^2}{\omega_n R} \int_{\partial B_R(0)} \frac{g(x)}{|x-y|^n} \, \mathrm{d}S(x). \tag{1.9}$$

The function

$$K(x,y) = -\partial_{\nu_x} G(x,y) = \frac{R^2 - |y|^2}{\omega_n R |x-y|^n}$$

is called the *Poisson kernel* for the ball $B_R(0)$ and formula (1.9) is called the Poisson integral formula for the ball $B_R(0)$.

THEOREM 3.2. *Given $g \in C^0\left(\partial B_R(0)\right)$, the function*

$$u(y) = \begin{cases} \displaystyle\int_{\partial B_R(0)} K(x,y) g(x) \, \mathrm{d}S(x), & y \in B_R(0), \\ g(y), & y \in \partial B_R(0), \end{cases}$$

*belongs to $C^2(B_R(0)) \cap C^0(\overline{B_R(0)})$ and is the unique solution of (1.7).*

PROOF. In the proof we use the shorthand notation $B_R = B_R(0)$. Since $G(x,y) = G(y,x)$ for all $x, y \in \overline{B_R}$, with $x \neq y$, the function $y \mapsto G(x,y)$ is harmonic in $B_R$ for any $x \in \partial B_R$, and so is $y \mapsto \partial_{\nu_x} G(x,y)$. It follows that $u$ is harmonic (and $C^\infty$) in $B_R$.

To prove that $u \in C^0\left(\overline{B_R}\right)$, we first observe that

$$\int_{\partial B_R} K(x,y) \, \mathrm{d}S(x) = 1, \qquad \text{for all } y \in B_R(0).$$

We want to show that, for all $y_0 \in \partial B_R$, we have $\lim_{B_R \ni y \to y_0} u(y) = g(y_0)$.

Since $g$ is continuous at $y_0$, we have: for all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that $|g(x) - g(y_0)| \leqslant \varepsilon$ for any $x \in \partial B_R$ such that $|x - y_0| \leqslant \delta_\varepsilon$. Then

$$\begin{aligned} |u(y) - g(y_0)| &= \left| \int_{\partial B_R} K(x,y) \left(g(x) - g(y_0)\right) \mathrm{d}S(x) \right| \\ &\leqslant \int_{\partial B_R} K(x,y) \left|g(x) - g(y_0)\right| \mathrm{d}S(x) \\ &= \underbrace{\int_{\partial B_R \cap B_{\delta_\varepsilon}(y_0)} K(x,y) \left|g(x) - g(y_0)\right| \mathrm{d}S(x)}_{=:\mathrm{A}} \\ &\qquad + \underbrace{\int_{\partial B_R \setminus B_{\delta_\varepsilon}(y_0)} K(x,y) \left|g(x) - g(y_0)\right| \mathrm{d}S(x)}_{=:\mathrm{B}}, \end{aligned}$$

where we have dropped the absolute value on the Poisson kernel since $K(x,y) > 0$ for $y \in B_R$ and $x \in \partial B_R$.

The first term can be bounded as

$$\mathrm{A} \leqslant \varepsilon \int_{\partial B_R \cap B_{\delta_\varepsilon}(y0)} K(x,y) \, \mathrm{d}S(x) < \varepsilon, \qquad \text{for all } y \in B_R$$

For the second term, we take $|y - y_0| \leqslant \tilde{\delta} < \delta_\varepsilon/2$ and $|x - y_0| \geqslant \delta_\varepsilon$. Then $|x - y| > \delta_\varepsilon/2$ and, since $|y_0| = R$, also $R - |y| \leqslant |y - y_0| \leqslant \tilde{\delta}$ so that

$$K(x,y) = \frac{R^2 - |y|^2}{\omega_n R |x-y|^n} \leqslant \frac{(R + |y|)(R - |y|) 2^n}{R \omega_n \delta_\varepsilon^n} < \frac{2R 2^n \tilde{\delta}}{R \omega_n \delta_\varepsilon^n}, \qquad \text{for all } x \notin \overline{B_{\delta_\varepsilon}(y_0)},\ y \in \overline{B_{\tilde{\delta}}(y_0)}$$

and, setting $M = \|g\|_{C^0(\partial B_R)}$, we have

$$\mathrm{B} < 2M \int_{\partial B_R \setminus B_{\delta_\varepsilon}(y_0)} K(x,y)\, \mathrm{d}S(x) \leqslant \frac{2^{n+2} M R^{n-1}}{\delta_\varepsilon^n} \tilde{\delta}.$$

Taking now $\tilde{\delta} < \min\left\{\frac{\delta_\varepsilon}{2}, \frac{\varepsilon \delta_\varepsilon^n}{2^{n+2} M R^{n-1}}\right\}$ we conclude that

$$|u(y) - g(y_0)| < 2\varepsilon \qquad \text{if } |y - y_0| < \tilde{\delta},$$

which proves the continuity of $u$ in $y_0$. ☐

**1.2. Dirichlet problem for the Laplace operator in a half-plane.** Let $\Omega := \mathbb{R}^n_+ = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_n > 0\}$. We want to build the Green function of the Laplacian in $\mathbb{R}^n_+$.

Again, the idea is to construct the function $h$ by suitable refleciton of $\Phi(\cdot - y)$. For any $y = (y_1, \cdots, y_{n-1}, y_n) \in \mathbb{R}^n_+$, we define the reflected point $\tilde{y} := (y_1, \cdots, y_{n-1}, -y_n) \notin \overline{\mathbb{R}^n_+}$.

We claim that $h(x,y) := -\Phi(x - \tilde{y})$ solves

$$\begin{cases} -\Delta h(x,y) = 0, & x \in \mathbb{R}^n_+, \\ h(x,y) = -\Phi(x-y), & x \in \partial \mathbb{R}^n_+, \end{cases}$$

and $G(x,y) := \Phi(x-y) - \Phi(x - \tilde{y})$ is the Green function of the Laplace equation in $\mathbb{R}^n_+$.

Moreover, since

$$\partial_{x_i} G(x,y) = \frac{1}{\omega_n}\left[-\frac{x_i - y_i}{|x-y|^n} + \frac{x_i - \tilde{y}_i}{|x - \tilde{y}|^n}\right] = \begin{cases} 0, & \text{if } i \neq n, \\ \frac{2y_n}{\omega_n |x-y|^n}, & \text{if } i = n, \end{cases}$$

the *Poisson Kernel* is

$$K(x,y) := -\partial_{\nu_x} G(x,y) = \partial_{x_n} G(x,y) = \frac{2y_n}{\omega_n |x-y|^n},$$

and a candidate solution to the Dirichlet problem in $\mathbb{R}^n_+$ is given by the following Poisson integral formula

$$u(y) = \int_{\partial \mathbb{R}^n_+} K(x,y) g(x)\, \mathrm{d}S(x).$$

We report these results in the following theorem (whose proof is left as an exercise in Exercise Sheet 6 for $n = 2$).

THEOREM 3.3. *Let $g \in C^0_c\left(\partial \mathbb{R}^n_+\right)$ and define*

$$u(y) := \begin{cases} \dfrac{2y_n}{\omega_n} \displaystyle\int_{\partial \mathbb{R}^n_+} \frac{g(x)}{|x-y|^n}\, \mathrm{d}x, & y \in \mathbb{R}^n_+, \\ g(y), & y \in \partial \mathbb{R}^n_+. \end{cases}$$

*Then, $u \in C^2\left(\mathbb{R}^n_+\right) \cap C^0\left(\overline{\mathbb{R}^n_+}\right)$ is the unique solution to the Dirichlet problem*

$$\begin{cases} -\Delta u = 0, & x \in \mathbb{R}^n_+, \\ u = g, & x \in \partial \mathbb{R}^n_+, \\ \lim_{|x| \to \infty} u(x) = 0. \end{cases}$$

**Lect. 7, 29.10**
**NB: 22.10 is in the semester break**

**1.3. Dirichlet problem for the Laplace operator in a general domain via Perron's method.** We now consider an open bounded set $\Omega \subseteq \mathbb{R}^n$ and the Dirichlet problem

$$\begin{cases} -\Delta u = 0, & x \in \Omega, \\ u = g, & x \in \partial\Omega. \end{cases} \tag{1.10}$$

In general, we will not be able to find a closed form expression for the Green function. However, we can still ask a more fundamental question whether the solution to such Dirichlet problem exists for any $g \in C^0(\partial\Omega)$. Observe that by the weak maximum principle, if a solution of this problem exists, it is unique (see Section 2.3.4).

If the answer to the existence question is positive, this will, in particular, ensure the existence of the Green function for the domain $\Omega$. To answer this question, we construct a solution using *Perron's method.*[1]

1.3.1. *Sub-harmonic functions and harmonic lifting.* As a first step, we revisit the characterization of sub-harmonic functions via the mean-value property by suitably generalizing Theorem 2.4.[2]

PROPOSITION 3.1. *Let $u \in C(\Omega)$. Then the following properties are equivalent.*

*(1) For all $x \in \Omega$ and $B_r(x) \subset\subset \Omega$,*

$$u(x) \leqslant \fint_{B_r(x)} u(y)\,\mathrm{d}y$$

*(2) For all $B \subset\subset \Omega$ and for all $h : \bar{B} \to \mathbb{R}$ which satisfies*

$$\begin{cases} -\Delta h = 0, & x \in B, \\ h \geqslant u, & x \in \partial B, \end{cases}$$

*we have $h(x) \geqslant u(x)$ for all $x \in \bar{B}$.*

*(3) For all $x \in \Omega$ and $B_r(x) \subset\subset \Omega$,*

$$u(x) \leqslant \fint_{\partial B_r(x)} u(y)\,\mathrm{d}S(y).$$

*(4) For all $x \in \Omega$ and for all $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local maximum in $x$, then $-\Delta\phi(x) \leqslant 0$.*

PROOF. <u>(1) $\implies$ (2).</u> We have that $u-h$ is a sub-harmonic function in $B$ such that $u-h \leqslant 0$ in $\partial B$. We conclude by the weak maximum principle.

<u>(2) $\implies$ (3).</u> Let $x \in \Omega$ and $B_r(x) \subset\subset \Omega$ and let $h$ be the solution to

$$\begin{cases} -\Delta h = 0, & y \in B_r(x), \\ h = u, & y \in \partial B_r(x). \end{cases}$$

Then by the Poisson integral formula (or by the property of spherical mean for harmonic functions) $h(x) = \fint_{\partial B_r(x)} h(y)\,\mathrm{d}S(y)$. Moreover by (2), $h(x) \geqslant u(x)$, so we conclude.

<u>(3) $\implies$ (1)</u>. Let $x \in \Omega$ and $B_r(x) \subset\subset \Omega$. Then, by the formula of integral over spheres and by (iii),

$$\int_{B_r(x)} u(y)\,\mathrm{d}y = \int_0^r \int_{\partial B_s(x)} u(y)\,\mathrm{d}S(y)\,\mathrm{d}s \geqslant \int_0^r u(x) n\omega_n s^{n-1}\,\mathrm{d}s = u(x)\omega_n r^n,$$

which gives the conclusion.

<u>(3) $\implies$ (4).</u> Let $x \in \Omega$, $\phi \in C^2(\Omega)$ and $B_r(x) \subseteq \Omega$ such that $u(y) - \phi(y) \leqslant u(x) - \phi(x)$ for all $y \in B_r(x)$. Since the inequality holds for every $y \in B_r(x)$ we get, by (3), that, for all $s \in (0, r)$,

$$u(x) - \phi(x) \geqslant \fint_{\partial B_s(x)} u(y)\,\mathrm{d}S(y) - \fint_{\partial B_s(x)} \phi(y)\,\mathrm{d}S(y) \geqslant u(x) - \fint_{\partial B_s(x)} \phi(y)\,\mathrm{d}S(y). \tag{1.11}$$

We define, for $s \in (0, r)$,

$$\psi(s) = \fint_{\partial B_s(x)} \phi(y)\,\mathrm{d}S(y) = \fint_{\partial B_1(0)} \phi(x + sz)\,\mathrm{d}S(z).$$

We note that $\lim_{s\to 0^+} \psi(s) = \phi(x)$ and so, owing to (1.11),

$$\phi(x) = \psi(0) \leqslant \psi(s), \qquad \text{for all } s \in (0, r). \tag{1.12}$$

Moreover, we compute, using the divergence theorem,

$$\psi'(s) = \fint_{\partial B_1(0)} D\phi(x + sz) \cdot z\,\mathrm{d}S(z) = \frac{1}{n\omega_n s^{n-1}} \int_{B_s(x)} \Delta\phi(y)\,\mathrm{d}y = \frac{s}{n} \fint_{B_s(x)} \Delta\phi(y)\,\mathrm{d}y.$$

Let us assume now by contradiction that (4) is not verified, then there exists $\delta > 0$ such that $-\Delta\phi(x) > 2\delta > 0$. By continuity, there exists $s > 0$ such that $-\Delta\phi(y) > \delta$ for all $y \in B_s(x)$. This

[1] Introduced by Oskar Perron [Per23].

[2] Point (iv) in Proposition 3.1 may be restated as "$u$ is sub-harmonic in $\Omega$ in the sense of viscosity solutions", where we refer to the theory of *viscosity solutions* introduced in [CL83; CEL84; CIL92].

gives that $\psi'(t) < -\delta t/n$ for all $t \leqslant s$, so $\psi(t) > \psi(s)$ for all $t \in (0,s)$ and then, integrating in $t \in (0,s)$, $\psi(s) - \psi(0) < -\frac{\delta s^2}{2n} < 0$, which contradicts (1.12).

(4) $\implies$ (3). Assume by contradiction that (3) is not verified. So there exists $x \in \Omega$ and there exists $r > 0$ such that

$$u(x) > \fint_{\partial B_r(x)} u(y)\,\mathrm{d}S(y).$$

Let us fix $c > 0$ sufficiently small such that

$$u(x) - \fint_{\partial B_r(x)} u(y)\,\mathrm{d}S(y) > c\,r^2. \tag{1.13}$$

Let $U \in C^2(B_r(x)) \cap C(\overline{B_r(x)})$ to be the unique solution to the Dirichlet problem

$$\begin{cases} -\Delta U = 0, & y \in B_r(x), \\ U(y) = u(y), & y \in \partial B_r(x). \end{cases}$$

Then, $U(y) = u(y)$ on $\partial B_r(x)$ and (by Poisson's integral formula) $U(x) = \fint_{\partial B_r(x)} u(y)\,\mathrm{d}S(y)$. Define

$$\phi(y) := U(y) + c\left(r^2 - |y-x|^2\right).$$

Then $u(y) - \phi(y) = 0$ if $y \in \partial B_r(x), \phi \in C^2(B_r(x)$ and $u(x) - \phi(x) = u(x) - U(x) - cr^2 > 0$ by the choice of $c$ in (1.13). Then $\max_{\overline{B_r(x)}} u(y) - \phi(y) > 0$ and there exists a point $z \in B_r(x)$ (we stress that the important thing is that $z$ is in the interior of $B_r(x)$) such that $u(z) - \phi(z) = \max_{\overline{B_r(x)}} u(y) - \phi(y)$. By (4), this implies that $-\Delta\phi(z) \leqslant 0$, but $\Delta\phi(z) = \Delta U(z) - c\Delta\left(|z-x|^2\right) = 0 - 2cn < 0$, and so we reached a contradiction. $\square$

We now define, given a sub-harmonic function $u$ in $\Omega$, its *harmonic lifting* in $B \subset \Omega$.

DEFINITION 3.2 (Harmonic lifting). *Let $u$ be a sub-harmonic function in $\Omega$ and $B \subset\subset \Omega$. Then the harmonic lifting of $u$ in $B$ is the function $U$ which coincides with $u$ in $\Omega \backslash B$ and in $B$ solves the Dirichlet problem*

$$\begin{cases} -\Delta U = 0, & x \in B, \\ U = u, & x \in \partial B. \end{cases}$$

REMARK 3.2. *By weak maximum principle, we have that $u \leqslant U$ in $\Omega$.*

REMARK 3.3. *Let $U$ be the harmonic lifting of $u$ in $B$, then $U$ is a sub-harmonic function. It is sufficient to show that $U$ satisfies property (2) in Proposition 3.1. Let $B' \subset\subset \Omega$ and $h$ be a function satisfying*

$$\begin{cases} -\Delta h = 0, & x \in B', \\ h \geqslant U, & x \in \partial B'. \end{cases}$$

*We consider two cases:*

*(1) if $B' \cap B = \varnothing$, then it is true that $h \geqslant u = U$ since $u$ is sub-harmonic.*
*(2) if $B' \cap B \neq \varnothing$, we split the domain into two parts: in $B' \backslash (B' \cap B)$, we have $h \geqslant u = U$ (arguing as in the previous case); in $B' \cap B$, we have that $h$ and $U$ are both harmonic, and moreover on $\partial\left(B' \cap B\right), h \geqslant U$, so we conclude by the weak maximum principle.*
*(3)*

1.3.2. *Existence result.* Let $\Omega$ be a bounded open set and $g \in L^\infty(\partial\Omega)$. Let us define

$$S_g := \{v \in C(\bar{\Omega}) : v \text{ sub-harmonic in } \Omega \text{ and } v(x) \leqslant g(x) \text{ for } x \in \partial\Omega\}.$$

REMARK 3.4. *The set $S_g$ is not empty and bounded from above. In fact the constant function $v := \inf_{\partial\Omega} g$ is in $S_g$. Moreover, by the weak maximum principle, we get $v \leqslant \sup_{\partial\Omega} g$ for all $v \in S_g$.*

THEOREM 3.4. *Let $\Omega$ be an open and bounded set and $g \in L^\infty(\partial\Omega)$. Then the function*

$$H_g(x) := \sup_{v \in S_g} v(x)$$

*is harmonic in $\Omega$.*

PROOF. Fix $x \in \Omega$. We want to show that $H_g$ is harmonic in $x$. Let $v_n$ be a sequence in $S_g$ such that $v_n(x) \to H_g(x)$.

Step 1. Without loss of generality, we can assume that $v_n$ is equi-bounded. Indeed, if it is not the case we consider the sequence $\tilde{v}_n = \max(v_n, \inf_{\partial\Omega} g)$. Note that $\tilde{v}_n \in S_g, \tilde{v}_n$ is equi-bounded (since $\inf_{\partial\Omega} g \leqslant \tilde{v}_n \leqslant \sup_{\partial\Omega} g$) and $\tilde{v}_n(x) \to H_g(x)$ (since $v_n(x) \leqslant \tilde{v}_n(x) \leqslant H_g(x)$). So $v_n$ is an equi-bounded sequence in $S_g$ with $v_n(x) \to H_g(x)$.

Step 2. We fix $r > 0$ such that $B_r(x) \subset\subset \Omega$ and consider for every $n$ the harmonic lifting $V_n$ of $v_n$ in $B_r(x)$. Then $V_n \in S_g$ and $V_n$ is equi-bounded (by the weak maximum principle and the fact that $v_n$ is equi-bounded) and $V_n(x) \to H_g(x)$. By Ascoli–Arzelà's theorem for harmonic functions (see Problem 1 in Exercise Sheet 7),[3] possibly passing to a subsequence (that we still denote with $V_n$) we get that $V_n \to V$ uniformly in $B_\rho(x)$ for every $\rho < r$. Moreover, we have that $V$ is harmonic in $B_r(x)$, $V(y) \leqslant H_g(y)$ for every $y \in B_r(x)$, and $V(x) = H_g(x)$.

Step 3. We claim now that there exists $\rho < r$ such that $V(y) = H_g(y)$ for every $y \in B_\rho(x)$. If it is true, we are done, since then $H_g$ is harmonic in $x$.

We assume that the claim is not true, so for every $\rho$ we find $z \in B_\rho(x)$ such that $V(z) < H_g(z)$. We prove that this leads to a contradiction.

Take a sequence $w_n \in S_g$ such that $w_n(z) \to H_g(z)$. As above, we can assume wlog that $w_n$ is equi-bounded. Moreover we can also assume that $w_n \geqslant V_n$ for every $n$. Indeed, if it is not the case we consider the sequence $\tilde{w}_n = \max(w_n, V_n)$. Note that $\tilde{w}_n \in S_g, \tilde{v}_n$ is equi-bounded (since $w_n$ and $V_n$ are equi-bounded) and $\tilde{w}_n(z) \to H_g(z)$ (since $w_n(z) \leqslant \tilde{w}_n(z) \leqslant H_g(z)$).

For every $n$ we consider the harmonic lifting $W_n$ of $w_n$ in $B_\rho(x)$. Then $W_n \in S_g, W_n$ is equi-bounded, $V_n(y) \leqslant W_n(y)$ (in particular $V_n(x) \leqslant W_n(x) \leqslant H_g(x)$). By Ascoli–Arzelà's theorem for harmonic functions, eventually passing to a subsequence (that we still denote with $W_n$) we get that $W_n \to W$ uniformly in $B_{\rho'}(x)$ for every $\rho' < \rho$. Moreover, $W$ is harmonic in $B_\rho(x)$, $V(y) \leqslant W(y)$ for every $y \in B_\rho(x), W(x) = H_g(x) = V(x)$ and $W(z) = H_g(z) > V(z)$.

So, $V, W$ are two harmonic functions in $B_\rho(x)$ such that $V - W \leqslant 0$, and $V(x) - W(x) = 0$. This implies, by the strong maximum principle, that $V \equiv W$ in $B_\rho(x)$, in contradiction with the fact that $W(z) > V(z)$. □

1.3.3. *Study of the boundary behaviour.* In Theorem 3.4, we proved that, for every bounded function $g$, there exists a harmonic function $H_g \in C^\infty(\Omega)$ which solves

$$\begin{cases} -\Delta H_g = 0, & x \in \Omega \\ H_g \geqslant g, & x \in \partial\Omega. \end{cases}$$

Now, we assume that $g \in C(\partial\Omega)$ and we wonder under which conditions $H_g$ is the solution of the Dirichlet problem (1.10), in particular under which conditions we can prove that, for all $x_0 \in \partial\Omega$,

$$\lim_{\substack{x \to x_0, \\ x \in \Omega}} H_g(x) = g(x_0). \tag{1.14}$$

Indeed, if we prove this identity, we get that $H_g \in C^\infty(\Omega) \cap C(\bar{\Omega})$ and coincides with $g$ on the boundary of $\Omega$.

REMARK 3.5. *Observe that in general we cannot expect that (1.14) holds true for every $\Omega$ bounded. Let $\Omega := \{x \in \mathbb{R}^2 : 0 < |x| < 1\}$ and $g \in C(\partial\Omega)$ such that $g(x) = 0$ for $|x| = 1$ and $g(0) = 1$. Then $H_g \equiv 0$ (and, in particular, it is not a solution of the Dirichlet problem with boundary data $g$ since $H_g(0) = 0 \neq 1$). In fact, $0 \in S_g$, so $H_g(x) \geqslant 0$ for every $x \in \Omega$. Let $v \in S_g$. So, by the weak maximum principle, $v(x) < 1$ for every $x \in \Omega$. Fix $\delta > 0$ and $\varepsilon = \varepsilon(\delta) \in (0,1)$ such that $-\delta\log(\varepsilon) > 1$. Consider now the function $w_\delta(x) = -\delta\log|x|$. This is harmonic in $\varepsilon < |x| < 1$, moreover $w_\delta(x) = 0$ if $|x| = 1$ and $w_\delta(x) = -\delta\log\varepsilon > 1$ if $|x| = \varepsilon$. This implies, by the weak maximum principle, that $v(x) \leqslant w_\delta(x)$ for every $\varepsilon \leqslant |x| \leqslant 1$. Moreover, $w_\delta(x) > 1 \geqslant v(x)$ also for every $|x| \leqslant \varepsilon$. Then $w_\delta(x) \geqslant v(x)$ for every $0 < |x| \leqslant 1$ and every $v \in S_g$, which implies that $H_g(x) \leqslant -\delta\log|x|$ for every $0 < |x| \leqslant 1$, and every $\delta > 0$, which gives the conclusion letting $\delta \to 0$.*

The continuity assumption (1.14) on the boundary is connected with the geometric properties of the boundary through the concept of barrier.

[3] Alternatively, we can define a new sequence $\{\tilde{V}_n := \max_{j \leqslant n} V_j\}_n$, which is now a non-decreasing sequence; we can then use Harnack's convergence theorem.

DEFINITION 3.3 (Local barrier). *Let $x_0 \in \partial\Omega$. Then $w$ is a* local barrier at $x_0$ *if there exists a neighborhood $U$ of $x_0$ such that $w \in C(\overline{\Omega \cap U})$ and*

*(1) $w$ is super-harmonic in $\Omega \cap U$;*
*(2) $w(x_0) = 0$ and $w(x) > 0$ for every $x \in \overline{\Omega \cap U} \setminus \{x_0\}$.*

*A* barrier $w$ *in $x_0$, is a local barrier with $U = \mathbb{R}^n$.*

REMARK 3.6. *Given a local barrier in $x_0$, it is always possible to construct a barrier in $x_0$ as follows: given $r > 0$ such that $B_{2r}(x_0) \subset\subset U$ and $m := \inf_{(U\setminus B_r(x_0))\cap\Omega} w > 0$, let*

$$w(x) := \begin{cases} \min\{m,\, w(x)\}, & x \in \overline{B(x_0, 2r) \cap \Omega}, \\ m, & \textit{elsewhere}. \end{cases}$$

*Then $w$ is a barrier in $x_0$.*

DEFINITION 3.4 (Regular points). *Let $x_0 \in \partial\Omega$. Then $x_0$ is a* regular point (with respect to the Laplacian)*, if there exists a (local) barrier at $x_0$.*

THEOREM 3.5. *Let $\Omega$ be an open bounded set, $g \in L^\infty(\partial\Omega)$. Let $x_0 \in \partial\Omega$. If $x_0$ is regular (with respect to the Laplacian) and $g$ is continuous in $x_0$, then (1.14) holds in $x_0$.*

PROOF. Fix $\varepsilon > 0$, then there exists $\delta > 0$ such that, for all $y \in \partial\Omega$, with $|y - x_0| \leqslant \delta$, we have $|g(y) - g(x_0)| \leqslant \varepsilon$ (since $g$ is continuous). Let $M := \|g\|_{C^0(\partial\Omega)}$. Let $w$ be a barrier in $x_0$. We have that $K := \min_{x\in\bar\Omega\setminus B_\delta(x_0)} w(x) > 0$. Let us define

$$\underline{w}(x) := g(\xi) - \epsilon - \frac{2M}{K} w(x), \quad \bar{w}(x) := g(\xi) + \epsilon + \frac{2M}{K} w(x).$$

We have that $\underline{w}$ is sub-harmonic and, for $x \in \partial\Omega$, $w(x) \leqslant g(x)$. Indeed,

$$\text{if } |x - \xi| < \delta, \quad \underline{w}(x) = \underbrace{g(\xi) - g(x) - \epsilon}_{<0} - \frac{2M}{K} w(x) + g(x) \leqslant g(x),$$

$$\text{if } |x - \xi| \geqslant \delta, \quad \underline{w}(x) \leqslant g(x) - \epsilon + \underbrace{g(\xi) - g(x) - \frac{2M}{K} w(x)}_{\leqslant 0 \text{ since } w(x) \geqslant K} \leqslant g(x).$$

Similarly, we can prove that $\bar{w}$ is super-harmonic and $\overline{w} \geqslant g(x)$ on $\partial\Omega$.

In conclusion, $\underline{w}(x) \leqslant H_g(x) \leqslant \bar{w}(x)$, for all $x \in \bar\Omega$, which implies

$$|H_g(x) - g(x_0)| \leqslant \epsilon + \frac{2M}{K} w(x).$$

Since $w(x) \to 0$ as $x \to x_0$, there exists $\tilde\delta > 0$ such that $w(x) < \frac{K}{2M}\epsilon$ for all $|x - x_0| \leqslant \tilde\delta$, and $|u(x) - g(x_0)| \leqslant 2\epsilon$ for all $x \in \bar\Omega$ such that $|x - x_0| \leqslant \tilde\delta$ which proves the continuity of $u$ in $x_0$. □

From Theorem 3.4 and Theorem 3.5 we get the following result.

COROLLARY 3.1. *For every $g \in C(\partial\Omega)$, the Dirichlet problem (1.10) admits a unique solution $u \in C^\infty(\Omega) \cap C(\bar\Omega)$ if and only if all the boundary points of $\Omega$ are regular.*

PROOF. Step 1. If $g$ is continuous and all the points of the boundary are regular, then $H_g$ is a solution of (1.10), and it is unique by weak maximum principle.

Step 2. If (1.10) admits a solution for every continuous boundary data, take $x_0 \in \partial\Omega$ and the solution $u$ to (1.10) with $g(x) = |x - x_0|$. Then the solution $u$ to (1.10) is a barrier in $x_0$. □

1.3.4. *Regular boundary points.* It remains open the question: for which domains $\Omega$ all the boundary points are regular? Sufficient conditions for this property to hold can be stated in terms of local geometric (for $n > 2$) or topological (for $n = 2$) properties of the boundary.[4]

We mention some of these conditions.

[4] The detailed study of this subject was initiated by Norbert Wiener [Wie24], and then extended to uniformly elliptic divergence-form equations with smooth coefficients by Werner Püschel [Püs32], and then to uniformly elliptic divergence form equations with bounded measureable coefficients by Walter Littman, Guido Stampacchia, and Hans Weinberger [LSW63]. We refer also to the useful book [Lan72] for further information and context. For fully-nonlinear equations (and, in particular, equations not in divergence form), see [LL23] and the references discussed therein.

Definition 3.5 (Exterior ball condition). *Let $\Omega$ be a open set of $\mathbb{R}^n$. We say that $\Omega$ has the exterior ball condition if, at every point $x \in \partial\Omega$, there exists $y \in \mathbb{R}^n \backslash \Omega$ and $r > 0$ such that $B_r(y) \subset \mathbb{R}^n \backslash \bar{\Omega}$ and $\overline{B_r(y)} \cap \bar{\Omega} = \{x\}$.*

Remark 3.7. *Observe that if $\Omega$ is convex, then the exterior ball condition is satisfied, due to the Hahn Banach separation theorem. Indeed at every point of $\partial\Omega$ it is possible to construct an hyperplane passing through that point and such that $\Omega$ is entirely contained in one of the two half spaces in which the space is divided by the hyperplane.*

*Observe that $\Omega$ of class $C^1$ is not sufficient to assure that the exterior ball condition is satisfied. For example, consider $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > x_1^2 \log|x_1|\}$. Then $\Omega$ is of class $C^1$ but in $(0,0)$ the exterior ball condition is not satisfied. Indeed to prove this, let $f(x) = x^2 \log|x|$ and $g(x) = \sqrt{r^2 - x^2} - r$, for $r > 0$ to be fixed. Note that $f(0) = g(0) = 0$. If the exterior ball condition were satisfied in $(0,0)$, then there would exist $r > 0$ such that $f(x) > g(x)$ for every $x \in (-r, r), x \neq 0$. But this is not the case, since $f'(x) < g'(x)$ for $x \to 0^+$ and $f'(x) > g'(x)$ for $x \to 0^-$.*

*If $\Omega$ is of class $C^2$, then we can show that it satisfies the exterior ball condition (also the interior ball condition).*

Proposition 3.2. *Let $\Omega$ be a bounded open set and $x_0 \in \partial\Omega$ such that in $x_0$ the exterior ball condition is satisfied. Then $x_0$ is regular (with respect to the Laplacian).*

Proof. Let $\Omega$ be a bounded open set and $x_0 \in \partial\Omega$ such that in $x_0$ it is satisfied the exterior ball condition: there exists $y_0 \in \mathbb{R}^n \backslash \Omega$ and $r_0 > 0$ such that $\bar{B}(y_0, r_0) \cap \bar{\Omega} = \{x_0\}$. Then $x_0$ is regular (with respect to the Laplacian).

It can be checked that a barrier is given by

$$w(x) := w(x) = \begin{cases} \frac{1}{r^{n-2}} - \frac{1}{|x-y_0|^{n-2}} & n > 2 \\ \log \frac{|x-y_0|}{r_0} & n = 2. \end{cases}$$

□

Definition 3.6 (Exterior cone condition). *Let $\Omega$ be a bounded open set. We say that $\Omega$ has the exterior cone condition if, at every point $x \in \partial\Omega$, there exists a cone[5] $C$ with $\operatorname{int} C \neq \varnothing$ and a neighborhood $U$ of $x$ such that such that $(x + C) \cap U \subset \mathbb{R}^n \backslash \Omega$.*

Proposition 3.3. *Let $\Omega$ be a bounded open set and $x_0 \in \partial\Omega$ such that in $x_0$ the exterior cone condition is satisfied. Then $x_0$ is regular (with respect to the Laplacian).*

Remark 3.8. *It is possible to prove that if $\Omega$ is of class $C^1$, then at every boundary point of $\Omega$ the exterior cone condition (and also the interior cone condition) is satisfied.*

*Actually, in order for the (exterior and interior) cone condition to be satisfied it is sufficient that the boundary of $\Omega$ is Lipschitz.*

Remark 3.9. *In dimension $n = 2$, much more irregular domains can be considered. It can be proved that $x_0 \in \partial\Omega$ is a regular boundary point if it is the endpoint of a single arc lying in the exterior of $\Omega$.*

## 2. Dirichlet problem for the Poisson equation

We consider now the Dirichlet problem for the Poisson equation

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ u = g, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

We could again take inspiration from the Green representation formula and construct a candidate solution as

$$u(y) = \int_\Omega G(x,y) f(x)\, \mathrm{d}x - \int_{\partial\Omega} \partial_{\nu_x} G(x,y) g(x)\, \mathrm{d}S(x),$$

where $G$ is the Green function of the Laplace equation for the domain $\Omega$. We know already by maximum principle that if a solution exists it is unique. However, for an arbitrary domain, we do not have, in general, the exact expression of the Green function. We proceed therefore in a slightly

[5] We recall that $C \subseteq \mathbb{R}^n$ is a *(convex) cone* if, for every $x, y \in C$, then $x + y \in C$ and $\lambda x \in C$ for every $\lambda > 0$.

different way. We consider first the function $w : \mathbb{R}^n \to \mathbb{R}$, called the *Newtonian potential of $f$*, given by

$$w(y) = \int_\Omega \Phi(x-y)f(x)\,\mathrm{d}x, \qquad y \in \mathbb{R}^n, \tag{2.2}$$

where $\Phi$ is the fundamental solution of the Laplace equation.

We will argue that, for a sufficiently smooth $f$, we have

$$w \in C^2(\Omega) \cap C^0(\bar{\Omega}), \quad \text{and} \quad -\Delta w = f \text{ in } \Omega; \tag{2.3}$$

next, we consider the problem

$$\begin{cases} \Delta u_0 = 0, & x \in \Omega, \\ u_0 = g - w, & x \in \partial\Omega, \end{cases}$$

which has a unique solution provided $\Omega$ is bounded with all boundary points regular and $g - w \in C^0(\partial\Omega)$, thanks to the results in Section 1.3. It follows that $u = u_0 + w$ is the unique solution of the original problem (2.1).

In summary, the sudy of the well-posedness of (2.1) boils down to showing that the Newtonian potential of $f$ satisfies (2.3).

Theorem 3.6. *Let $\Omega$ be a bounded domain and $f \in C^2_c(\Omega)$. Then, the Newtonian potential (2.2) of $f$ satisfies $w \in C^2(\mathbb{R}^n)$ and $-\Delta w = f$ in $\Omega$.*

Proof. Since $f$ has compact support in $\Omega$, its extension by zero outside $\Omega$, which we denote $\tilde{f}$, is still a $C^2$ function. We can then rewrite the Newtonian potential as

$$w(y) = \int_\Omega \Phi(x-y)f(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \Phi(x-y)\tilde{f}(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \Phi(z)\tilde{f}(z+y)\,\mathrm{d}z.$$

Since $\tilde{f} \in C^2_c(\mathbb{R}^n)$ and $\Phi$ is locally integrable we have that $w \in C^2(\mathbb{R}^n)$ and

$$-\Delta w(y) = -\int_{\mathbb{R}^n} \Phi(z)\Delta\tilde{f}(z+y)\,\mathrm{d}z = -\int_{B_R(0)} \Phi(x-y)\Delta\tilde{f}(x)\,\mathrm{d}x,$$

where $B_R(0)$ is any sufficiently large ball that contains $\Omega$. Green's representation formula yields

$$\begin{aligned} \tilde{f}(y) = &\int_{B_R(0)} \Phi(x-y)(-\Delta\tilde{f}(x))\,\mathrm{d}x \\ &- \int_{\partial B_R(0)} \partial_\nu\Phi(x-y)\tilde{f}(x)\,\mathrm{d}S(x) + \int_{\partial B_R(0)} \Phi(x-y)\partial_\nu\tilde{f}(x)\,\mathrm{d}S(x); \end{aligned}$$

since $\tilde{f} \in C^2_c(B_R(0))$, the boundary terms vanish and we conclude that $-\Delta w(y) = \tilde{f}(y) = f(y)$ for all $y \in \Omega$. ☐

Corollary 3.2. *Let $\Omega$ be a bounded domain with all boundary points regular, $g \in C^0(\partial\Omega)$ and $f \in C^2_0(\Omega)$. Then there exists a unique solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ of (2.1).*

Remark 3.10. *One could generalize the theorems above and show that, if $f \in C^{0,\alpha}$, the Newtonian potential satisfies $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and $-\Delta w = f$. If $\Omega$ is a bounded domain with all boundary points regular and $g \in C^0(\partial\Omega)$, then the Dirichlet problem for the Poisson equation admits a unique solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and, moreover, $u \in C^{2,\alpha}(\Omega)$.*

*Here, we recall that a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is* locally Hölder continuous in $\Omega$ with exponent $\alpha \in (0,1]$ *if, in any compact set $K \subset \Omega$,*

$$|f|_{\alpha,K} := \sup_{\substack{x,\,y\in K,\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty;$$

*and $C^{0,\alpha}(\Omega)$ denotes the space of locally $\alpha$-Hölder continuous functions in $\Omega$.*

CHAPTER 4

# Sobolev spaces and weak solutions to second-order elliptic PDEs

## 1. Sobolev spaces

**1.1.** $L^p$ **spaces.** Let $\Omega \subset \mathbb{R}^n$ be a domain and denote by $L^1(\Omega)$ the space of Lebesgue integrable functions from $\Omega$ to $\mathbb{R}$, where, as usual, two functions are identified if they coincide almost everywhere, i.e., $L^1(\Omega)$ is the space of equivalence classes where two functions are equivalent if the set where they differ has zero Lebesgue measure. **Lect. 8, 5.11**

We recall the definition of $L^p$ spaces.[1]

DEFINITION 4.1 ($L^p$ spaces). *For $1 \leqslant p < \infty$, we define*

$$L^p(\Omega) := \{f : \Omega \to \mathbb{R},\ \textit{measurable, s.t. } |f|^p \in L^1(\Omega)\}$$

*and, for $p = \infty$,*

$$L^\infty(\Omega) := \{f : \Omega \to \mathbb{R} \textit{ measurable, s.t. there exists } C > 0 \textit{ s.t. } |f| \leqslant C \textit{ a.e. in } \Omega\}$$

*For $p \in [1, \infty]$, we denote define*

$$L^p_{loc}(\Omega) = \{f : \Omega \to \mathbb{R} \textit{ s.t. } f \in L^p(K) \textit{ for any } K \subset\subset \Omega\}.$$

*For a function $f \in L^p(\Omega)$ and $p \in [1, \infty]$, we denote*

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p \,\mathrm{d}x\right)^{1/p}, \quad 1 \leqslant p < \infty,$$
$$\|f\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| := \inf\{C : |f| \leqslant C \textit{ a.e. in } \Omega\}.$$

We recall (without proof) some fundamental properties of $L^p$ spaces.

**Completedness:** With the definitions above, the normed vector space $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ is a Banach space[2] for any $1 \leqslant p \leqslant \infty$, i.e., every Cauchy sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^p(\Omega)$ has a limit in $L^p(\Omega)$. We recall, moreover, that every Cauchy sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^p(\Omega)$ has a sub-sequence converging pointwise a.e. in $\Omega$. For $p = 2$, the space $L^2(\Omega)$ is a Hilbert space[3] with inner product $(f, g)_{L^2(\Omega)} := \int_\Omega fg$.

**Hölder inequality:** [4] Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, with $1/p + 1/q = 1$. Then $fg \in L^1(\Omega)$ and

$$\int_\Omega f(x)g(x)\,\mathrm{d}x \leqslant \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}$$

More generally, if $f_i \in L^{p_i}(\Omega)$, for $i = 1, \cdots, r$ with $\sum_{i=1}^r 1/p_i = 1$, then $\prod_{i=1}^r f_i \in L^1(\Omega)$ and

$$\int_\Omega \prod_{i=1}^r f_i(x)\,\mathrm{d}x \leqslant \prod_{i=1}^r \|f_i\|_{L^{p_i}(\Omega)}$$

**Embeddings:** Let $\Omega$ be bounded. For any $1 \leqslant p \leqslant q < \infty$, if $f \in L^q(\Omega)$, then $f \in L^p(\Omega)$ and

$$\|f\|_{L^p(\Omega)} \leqslant |\Omega|^{1/p-1/q}\|f\|_{L^q(\Omega)}.$$

If $f \in L^\infty(\Omega)$, then $f \in L^p(\Omega)$ for all $1 \leqslant p < \infty$ and $\|f\|_{L^p(\Omega)} \leqslant |\Omega|^{1/p}\|f\|_{L^\infty(\Omega)}$. Moreover, $\lim_{p\to\infty} \|f\|_{L^p(\Omega)} = \|f\|_{L^\infty(\Omega)}$.

[1] Sometimes also called *Lebesgue spaces*, after Henri Lebesgue, although they were first introduced by Frigyes Riesz [Rie10] (see [Pie07, Section 1.1.4] for further historical discussions).

[2] Named after Stefan Banach (see [Pie07, Chapter 1] for further historical information).

[3] Named after David Hilbert (see [Pie07, Chapter 1, Section 1.5]).

[4] Named after Otto Hölder [Höl89], but previously proven by Leonard James Rogers [Rog88]. See [Mal98].

Thus, the embedding $L^p(\Omega) \hookrightarrow L^q(\Omega)$ is continuous. This result is not true if the measure of $\Omega$ is infinite, but in general we have the following interpolation result: if $1 \leqslant p \leqslant q \leqslant r$, then $L^p(\Omega) \cap L^r(\Omega) \hookrightarrow L^q(\Omega)$ and

$$\|f\|_q \leqslant \|f\|_p^\theta \|f\|_r^{1-\theta},$$

where $0 \leqslant \theta \leqslant 1$ is given by $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}$.

**Dual of $L^p$ and reflexivity:** We recall that the (topological) dual of a Banach space $(V, \|\cdot\|_V)$, denoted $V'$, is the space of linear and bounded functionals $G : V \to \mathbb{R}$. It is also a Banach space when endowed with the norm $\|G\|_{V'} := \sup_{\substack{f \in V \\ f \neq 0}} \frac{|G(f)|}{\|f\|_V}$. For $1 \leqslant p < \infty$, the dual space of $L^p(\Omega)$ can be identified with $L^q(\Omega)$ with $q = \frac{p}{p-1}$: for any functional $G \in (L^p(\Omega))'$, there exists a unique function $g \in L^q(\Omega)$ such that

$$G(f) = \int_\Omega g(x) f(x) \, \mathrm{d}x, \qquad \text{for all } f \in L^p(\Omega).$$

Moreover, $\|G\|_{(L^p(\Omega))'} = \|g\|_{L^q(\Omega)}$. Thanks to this identification, the spaces $L^p(\Omega)$ for $p \in (1, \infty)$ are reflexive, i.e., the "double dual" $(L^p(\Omega))''$ can be identified with $L^p(\Omega)$ itself. This property does not hold for $p = 1, \infty$ and, in fact, $(L^\infty(\Omega))' \supset L^1(\Omega)$ with strict inclusion.

**Density of continuous functions and separability:** $C_c^0(\Omega)$ is dense in $L^p(\Omega)$ for any $1 \leqslant p < \infty$, i.e., for any $f \in L^p(\Omega)$ and $\epsilon > 0$, there exists $g \in C_c^0(\Omega)$ such that $\|f - g\|_{L^p(\Omega)} \leqslant \epsilon$. Such density result is not true, however, in $L^\infty(\Omega)$. From this result it also follows that all $L^p(\Omega)$ spaces with $1 \leqslant p < \infty$ are separable, i.e., they have a countable dense subspace.[5]

**Mollification:** Let $\eta_\epsilon$ denote the standard mollifier and, for any $f \in L^1_{\mathrm{loc}}(\Omega)$ and $\epsilon > 0$, denote by $f_\epsilon : \Omega_\epsilon \to \mathbb{R}$ the $\epsilon$-mollification of $f$:

$$f_\epsilon(x) = (\eta_\epsilon * f)(x) = \int_\Omega \eta_\epsilon(x - y) f(y) \, \mathrm{d}y, \qquad x \in \Omega_\epsilon,$$

with $\Omega_\epsilon := \{y \in \Omega : \operatorname{dist}(y, \partial\Omega) > \epsilon\}$. Then $f_\epsilon \in C^\infty(\Omega_\epsilon)$ for any $\epsilon > 0$. Moreover, if $f \in L^p_{\mathrm{loc}}(\Omega)$, with $1 \leqslant p < \infty$, for any $V \subset\subset \Omega$, we have $\|f_\epsilon\|_{L^p(V)} \leqslant \|f\|_{L^p(V)}$ and $\lim_{\epsilon \to 0} \|f - f_\epsilon\|_{L^p(V)} = 0$.

**Density of $C_c^\infty(\Omega)$:** Using $\epsilon$-mollification, we can show that $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, for any $1 \leqslant p < \infty$.

**Fundamental lemma of the calculus of variations:** Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in L^1_{\mathrm{loc}}(\Omega)$. If $\int_\Omega f\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$, then $f = 0$ a.e. in $\Omega$.

**1.2. Weak derivatives and Sobolev spaces.** In Section 3, we presented the notions of distribution and of distributional derivative of a distribution (which always exist). Here we present the notion of weak derivative, where we require some additional integrability.

Definition 4.2 (Weak derivative). *Let $u \in L^1_{loc}(\Omega)$ and $\alpha \in \mathbb{N}^n$ be a multi-index. We say that $u$ has $\alpha^{th}$ weak derivative if its distributional derivative satisfies $D^\alpha u \in L^1_{loc}(\Omega)$, i.e., there exists $v \in L^1_{loc}(\Omega)$such that*

$$\int_\Omega v\phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi \, \mathrm{d}x, \qquad \textit{for all } \phi \in \mathcal{D}(\Omega).$$

Lemma 4.1 (Uniqueness of weak derivative). *An $\alpha^{th}$-weak derivative of $u \in L^1_{loc}(\Omega)$, if it exists, is uniquely defined up to a zero measure set.*

Proof. Suppose that there exist two $\alpha^{\mathrm{th}}$-weak derivatives $v, \tilde{v} \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega v\phi = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi = \int_\Omega \tilde{v}\phi, \qquad \text{for all } \phi \in \mathcal{D}(\Omega).$$

Then $\int_\Omega (v - \tilde{v})\phi = 0$ for all $\phi \in \mathcal{D}(\Omega)$, which implies $v = \tilde{v}$ a.e. in $\Omega$ by the fundamental lemma of the calculus of variations. □

[5] To show this, it is enough to take a sequence of compact subdomains $\Omega_m \subset\subset \Omega$ such that $\Omega = \bigcup_{m=1}^\infty \Omega_m$ and the spaces $P_m$ of polynomials in $\Omega_m$, extended by zero on $\Omega_m^c$, having rational coefficients. Then $P = \bigcup_{m=1}^\infty P_m$ is countable and dense in $C_c^0(\Omega)$ hence in $L^p(\Omega)$ as well.

REMARK 4.1. *If $u \in C^k(\Omega)$, then the classical $\alpha$-th derivatives $D^\alpha u, |\alpha| \leqslant k$, coincide with the $\alpha$-th weak derivatives. Indeed, if we denote by $v_\alpha \in L^1_{loc}(\Omega)$ the $\alpha$-th weak derivative, we have*

$$\int_\Omega v_\alpha \phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi \, \mathrm{d}x = \int_\Omega D^\alpha u \phi \, \mathrm{d}x, \qquad \textit{for all } \phi \in \mathcal{D}(\Omega),$$

*which implies $D^\alpha u = v_\alpha$ a.e. in $\Omega$ and, since $D^\alpha u$ is continuous, $v_\alpha$ admits a continuous version coinciding with $D^\alpha u$. Hence, the notion of weak-derivative extends that of classical derivative and, for this reason, we use the same symbol $D^\alpha u$ to denote either classical or weak (or distributional) derivative depending on the context.*

The next lemma extends another familiar result, stating that the weak derivative of a limit coincides with the limit of the weak derivatives.

LEMMA 4.2 (Convergence of weak derivatives). *Consider a sequence of functions $f_n \in L^1_{\mathrm{loc}}(\Omega)$. For a fixed multi-index $\alpha$, assume that each $f_n$ admits the weak derivative $g_n = D^\alpha f_n$. If $f_n \to f$ and $g_n \to g$ in $L^1_{\mathrm{loc}}(\Omega)$, then $g = D^\alpha f$.*

PROOF. For every test function $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$, a direct computation yields

$$\begin{aligned}\int_\Omega g\phi \, \mathrm{d}x &= \lim_{n\to\infty} \int_\Omega g_n \phi \, \mathrm{d}x = \lim_{n\to\infty} (-1)^{|\alpha|} \int_\Omega f_n D^\alpha \phi \, \mathrm{d}x \\ &= (-1)^{|\alpha|} \int_\Omega f D^\alpha \phi \, \mathrm{d}x\end{aligned}$$

By definition, this means that $g$ is the $\alpha$-th weak derivative of $f$. □

We are now ready to define *Sobolev spaces*.[6]

DEFINITION 4.3 (Sobolev spaces). *For $1 \leqslant p \leqslant \infty$, we define*

$$W^{k,p}(\Omega) := \{ f \in L^p(\Omega) : \ D^\alpha f \in L^p(\Omega), \ \textit{for all } \alpha \in \mathbb{N}^n, |\alpha| \leqslant k \} \,.$$

*For $p = 2$, the space $W^{k,2}(\Omega)$ is denoted by $H^k(\Omega)$ and, for $k = 0$, $W^{0,p}(\Omega) = L^p(\Omega)$. Finally, we define*

$$W^{k,p}_{loc} := \left\{ f : \Omega \to \mathbb{R} \ \textit{s.t.} \ f \in W^{k,p}(K) \ \textit{for all } K \subset\subset \Omega \right\}.$$

*On the space $W^{k,p}(\Omega)$, we define the semi-norms*[7]

$$\begin{aligned}|f|_{W^{k,p}(\Omega)} &:= \left( \sum_{|\alpha|=k} \int_\Omega |D^\alpha f|^p \right)^{1/p}, \qquad \textit{for } 1 \leqslant p < \infty, \\ |f|_{W^{k,\infty}(\Omega)} &:= \max_{|\alpha|=k} \sup_\Omega D^\alpha f,\end{aligned}$$

*and the norms*

$$\begin{aligned}\|f\|_{W^{k,p}(\Omega)} &:= \left( \sum_{|\alpha|\leqslant k} \int_\Omega |D^\alpha f|^p \right)^{1/p} \\ &\simeq \sum_{|\alpha|\leqslant k} \|D^\alpha f\|_{L^p(\Omega)} \simeq \max_{\alpha \leqslant k} \|D^\alpha f\|_{L^p(\Omega)}, \qquad \textit{for } 1 \leqslant p < \infty, \\ \|f\|_{W^{k,\infty}(\Omega)} &:= \max_{|\alpha|\leqslant k} \sup_\Omega D^\alpha f \simeq \sum_{\|\alpha|\leqslant k} \|D^\alpha f\|_{L^\infty(\Omega)}.\end{aligned}$$

We recall that two norms $\|\cdot\|$ and $|\cdot|$ on a vector space $X$ are equivalent if there exist constants $c_1, c_2 \in (0, \infty)$ such that

$$\|x\| \leqslant c_1 |x| \leqslant c_2 \|x\| \qquad \text{for all } x \in X$$

In this case, we often write (as above) $\|\cdot\| \simeq |\cdot|$. Note that the property of a set to be open, closed, compact, or complete in a normed space is not affected if the norm is replaced by an equivalent norm.

We will often use the shorthand notation $\|\cdot\|_{k,p,\Omega}$ or simply $\|\cdot\|_{k,p}$ for $\|\cdot\|_{W^{k,p}(\Omega)}$, if no ambiguity arises.

[6] Named after Sergei Sobolev [Sob91] (see also the discussion in [Nau02]).

[7] A seminorm $q$ on a vector space has all the properties of a norm except that $q(f) = 0$ need not imply $f = 0$.

LEMMA 4.3. *The application* $\|\cdot\|_{k,p} : W^{k,p}(\Omega) \to \mathbb{R}_+$ *is a norm for any* $1 \leqslant p \leqslant \infty$.

PROOF. The proof relies on the fact that $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{\ell^p(\mathbb{R}^m)}$ are norms. Indeed, clearly, $\|\lambda f\|_{k,p} = |\lambda|\|f\|_{k,p}$ for any $\lambda \in \mathbb{R}$; moreover, $\|f\|_{k,p} = 0$ implies, in particular, $\|f\|_{L^p} = 0$, hence $f = 0$ a.e. in $\Omega$. Concerning the triangular inequality, for all $f, g \in W^{k,p}(\Omega)$ and $1 \leqslant p < \infty$, we have

$$
\begin{aligned}
\|f+g\|_{k,p} &= \left( \sum_{|\alpha| \leqslant k} \|D^\alpha f + D^\alpha g\|_{L^p}^p \right)^{1/p} \\
&\leqslant \left( \sum_{|\alpha| \leqslant k} (\|D^\alpha f\|_{L^p} + \|D^\alpha g\|_{L^p})^p \right)^{1/p} \\
&\leqslant \left( \sum_{|\alpha| \leqslant k} \|D^\alpha f\|_{L^p}^p \right)^{1/p} + \left( \sum_{|\alpha| \leqslant k} \|D^\alpha g\|_{L^p}^p \right)^{1/p} \\
&= \|f\|_{k,p} + \|g\|_{k,p}.
\end{aligned}
$$

Similarly, for $p = \infty$, we have

$$
\begin{aligned}
\|f+g\|_{k,\infty} &= \max_{|\alpha| \leqslant k} \|D^\alpha f + D^\alpha g\|_{L^\infty(\Omega)} \\
&\leqslant \max_{|\alpha| \leqslant k} \left( \|D^\alpha f\|_{L^\infty(\Omega)} + \|D^\alpha g\|_{L^\infty(\Omega)} \right) \leqslant \|f\|_{k,\infty} + \|g\|_{k,\infty}
\end{aligned}
$$

Hence $\|\cdot\|_{k,p}$ satisfies all the properties of a norm. □

LEMMA 4.4. *The normed vector space* $\left(W^{k,p}(\Omega), \|\cdot\|_{k,p}\right)$ *is a Banach space for every* $k \in \mathbb{N}$ *and* $1 \leqslant p \leqslant \infty$. *In particular, the space* $H^k(\Omega) = W^{k,2}(\Omega)$ *is a Hilbert space, for every* $k \in \mathbb{N}$, *with inner product*

$$
(f, g)_{H^k} := \sum_{|\alpha| \leqslant k} \int_\Omega D^\alpha f \cdot D^\alpha g \,\mathrm{d}x.
$$

PROOF. Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $W^{k,p}(\Omega)$, i.e., for all $\epsilon > 0$, there exists $N > 0$ such that

$$
\|f_n - f_m\|_{k,p} = \left( \sum_{|\alpha| \leqslant k} \|D^\alpha f_n - D^\alpha f_m\|^p \right)^{1/p} \leqslant \epsilon, \qquad \text{for all } n, m \geqslant N.
$$

As a consequence, for each $\alpha \in \mathbb{N}^n$, $|\alpha| \leqslant k$, the sequence $\{D^\alpha f_n\}_{n=1}^\infty$ is Cauchy in $L^p(\Omega)$. Since $L^p(\Omega)$ is complete, there exists $f_\alpha \in L^p$ such that $D^\alpha f_n \to f_\alpha$ in $L^p(\Omega)$. In particular, for $\alpha = (0, \cdots, 0)$, $f_n \xrightarrow{L^p} f_{(0,\cdots,0)} =: f$. We claim that $f_\alpha = D^\alpha f$, for all $|\alpha| \leqslant k$. Indeed,

$$
\int_\Omega f D^\alpha \phi = \lim_{n\to\infty} \int_\Omega f_n D^\alpha \phi = \lim_{n\to\infty} (-1)^{|\alpha|} \int_\Omega D^\alpha f_n \phi = (-1)^{|\alpha|} \int_\Omega f_\alpha \phi, \qquad \text{for all } \phi \in \mathcal{D}(\Omega).
$$

The fact that one can exchange the limit and integration is just a consequence of Hölder inequality since $D^\alpha \phi \in L^q(\Omega)$ with $1/q + 1/p = 1$ and $\left|\int f D^\alpha \phi - \int f_n D^\alpha \phi\right| \leqslant \|f - f_n\|_{L^p} \|D^\alpha \phi\|_{L^q} \xrightarrow{n\to\infty} 0$. The same argument applies to show that $\lim_{n\to\infty} \int_\Omega D^\alpha f_n \phi = \int_\Omega f_\alpha \phi$. Therefore, $D^\alpha f_n \xrightarrow{L^p} D^\alpha f$, for all $|\alpha| \leqslant k$, implying that $f_n \to f$ in $W^{k,p}$.

Finally, the bilinear form $(\cdot,\cdot)_{H^k}$ induces the norm $\|\cdot\|_{k,2}$ and it is immediate to verify that it satisfies all the assumptions of an inner product (thanks to the fact that $\int_\Omega f \cdot g$ is an inner product in $L^2(\Omega)$). □

REMARK 4.2. *The inner product in Lemma 4.4 induces the norm*

$$
\|f\|_{H^k(\Omega)} = \left( \sum_{|\alpha| \leqslant k} \int_\Omega |D^\alpha f|^2 \right)^{1/2},
$$

*while the equivalent norm* $\sum_{|\alpha| \leqslant k} \|D^\alpha f\|_{L^2(\Omega)}$ *is not induced by a scalar product.*

LEMMA 4.5 (Properties of weak derivatives). *Let $f, g \in W^{k,p}(\Omega)$ and $\alpha \in \mathbb{N}^n, |\alpha| \leqslant k$. The following properties of the $\alpha^{th}$-weak derivatives hold.*

**Linearity:** *For all $\lambda, \mu \in \mathbb{R}$, we have $\lambda f + \mu g \in W^{k,p}(\Omega)$ and*

$$D^\alpha(\lambda f + \mu g) = \lambda D^\alpha f + \mu D^\alpha g.$$

**Commutativity:** *For all $\alpha, \beta \in \mathbb{N}^n$ such that $|\alpha| + |\beta| \leqslant k$, we have*

$$D^\beta \left(D^\alpha f\right) = D^\alpha \left(D^\beta f\right) = D^{\alpha+\beta} f.$$

**Leibniz's[8] rule:** *For all $\xi \in C^\infty(\Omega)$, we have $\xi f \in W^{k,p}_{loc}(\Omega)$ and*

$$D^\alpha(\xi f) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^\beta \xi D^{\alpha-\beta} f.$$

LEMMA 4.6. *Let $f \in W^{k,p}(\Omega)$, with $1 \leqslant p < \infty$, and $f_\epsilon := \eta_\epsilon * f : \Omega \to \mathbb{R}$. Then $f_\epsilon \xrightarrow{\epsilon\to 0} f$ in $L^p(\Omega)$ and $f_\epsilon \xrightarrow{\epsilon\to 0} f$ in $W^{k,p}(K)$, for any $K \subset\subset \Omega$.*

PROOF. We know already that $f_\epsilon \to f$ in $L^p(\Omega)$ when $\epsilon \to 0$, so we are left to prove that $D^\alpha f_\epsilon \xrightarrow{\epsilon\to 0} D^\alpha f$ in $L^p(K)$ for any $\alpha \in \mathbb{N}^n$, $|\alpha| \leqslant k$ and $K \subset\subset \Omega$. We first show that $D^\alpha f_\epsilon = \eta_\epsilon * D^\alpha f$ in $\Omega_\epsilon := \{y \in \Omega,\ \mathrm{dist}(y, \partial\Omega) > \epsilon\}$, where $D^\alpha f_\epsilon$ is a classical derivative since $f_\epsilon \in C^\infty(\Omega_\epsilon)$, whereas $D^\alpha f$ is a weak derivative. We have indeed $\forall x \in \Omega_\epsilon$

$$\begin{aligned}
D^\alpha f_\epsilon(x) &= D^\alpha \int_\Omega \eta_\epsilon(x-y) f(y) dy = \int_\Omega D_x^\alpha \eta_\epsilon(x-y) f(y) dy \\
&= (-1)^{|\alpha|} \int_\Omega D_y^\alpha \underbrace{\eta_\epsilon(x-y)}_{\in \mathcal{D}(\Omega)} f(y) dy \\
&= \int_\Omega \eta_\epsilon(x-y) D^\alpha f(y) dy = \left(\eta_\epsilon * D^\alpha f\right)(x).
\end{aligned}$$

It follows that $D^\alpha f_\epsilon \to D^\alpha f$ in $L^p(K)$, for all $K \subset\subset \Omega$. Hence,

$$\|f - f_\epsilon\|_{W^{k,p}(K)} = \left(\sum_{|\alpha| \leqslant k} \|D^\alpha f - D^\alpha f_\epsilon\|^p_{L^p(K)}\right)^{\frac{1}{p}} \xrightarrow{\epsilon\to 0} 0.$$

□

EXAMPLE 4.1. *Consider $\Omega = (0,1) \subset \mathbb{R}$ and the constant function $f = 1$ in $\Omega$, whose (classical/weak) derivative is $f' = 0$ in $\Omega$ and clearly $f \in W^{1,p}(\Omega)$ for any $1 \leqslant p \leqslant \infty$. Let $f_\epsilon(x) = \int_0^1 \eta_\epsilon(x-y) f(y)\,\mathrm{d}y$, whose derivative is*

$$f'_\epsilon(x) = \int_0^1 \partial_x \eta_\epsilon(x-y) f(y)\,\mathrm{d}y = \int_0^1 -\partial_y \eta_\epsilon(x-y)\,\mathrm{d}y = \eta_\epsilon(x) - \eta_\epsilon(x-1).$$

*We notice that, for $\epsilon < \frac{1}{2}$, we have $|\eta_\epsilon(x) - \eta_\epsilon(x-1)| = \eta_\epsilon(x) + \eta_\epsilon(x-1)$ for all $x \in [0,1]$.*

*We claim that $f'_\epsilon \nrightarrow 0$ in $L^p(0,1)$ for any $p \geqslant 1$. Indeed,*

$$\|f'_\epsilon\|_{L^p((0,1))} \geqslant \|f'_\epsilon\|_{L^1((0,1))} = \int_0^1 \eta_\epsilon(x)\,\mathrm{d}x + \int_0^1 \eta_\epsilon(x-1)\,\mathrm{d}x = 1.$$

*In conclusion, $f_\epsilon \nrightarrow f$ in $W^{1,p}(0,1)$ for any $p \geqslant 1$. However, we have $f'_\epsilon \to 0$ in $L^p(K)$ for any $K \subset\subset (0,1)$.*

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**1.3. Approximation of Sobolev functions.** To begin with, we consider Sobolev functions defined on all of $\mathbb{R}^n$. They may be approximated in the Sobolev norm by test functions.

THEOREM 4.1. *For $k \in \mathbb{N}$ and $1 \leqslant p < \infty$, the space $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$*

PROOF. We will prove the result in two steps.

Step 1. First, we show that $C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$.

Let $\eta^\epsilon \in C_c^\infty(\mathbb{R}^n)$ be the standard mollifier and $f \in W^{k,p}(\mathbb{R}^n)$. Then $\eta^\epsilon * f \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ and, for $|\alpha| \leqslant k$,

$$D^\alpha(\eta^\epsilon * f) = \eta^\epsilon * (D^\alpha f) \to D^\alpha f \quad \text{in } L^p(\mathbb{R}^n) \text{ as } \epsilon \to 0^+.$$

It follows that $\eta^\epsilon * f \to f$ in $W^{k,p}(\mathbb{R}^n)$ as $\epsilon \to 0$.

Step 2. Now we prove that $C_c^\infty(\Omega)$ is dense in $W^{k,p}(\mathbb{R}^n)$. Let us suppose that $f \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$, and let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function such that

$$\phi(x) = \begin{cases} 1, & \text{if } |x| \leqslant 1, \\ 0, & \text{if } |x| \geqslant 2. \end{cases}$$

Define $\phi^R(x) = \phi(x/R)$ and $f^R = \phi^R f \in C_c^\infty(\mathbb{R}^n)$. Then, by the Leibnitz rule,

$$D^\alpha f^R = \phi^R D^\alpha f + \frac{1}{R} h^R,$$

where $h^R$ is a function that is bounded in $L^p$ uniformly in $R$. Hence, by Lebesgue's dominated convergence theorem

$$D^\alpha f^R \to D^\alpha f \quad \text{in } L^p \text{ as } R \to \infty,$$

so $f^R \to f$ in $W^{k,p}(\mathbb{R}^n)$ as $R \to \infty$. This concludes the proof. □

If $\Omega$ is a proper open subset of $\mathbb{R}^n$, then $C_c^\infty(\Omega)$ is not dense in $W^{k,p}(\Omega)$. Instead, its closure is the space of functions $W_0^{k,p}(\Omega)$ that "vanish on the boundary $\partial\Omega$". We discuss this further in Section 1.5. The space $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$ for any open set $\Omega$ (cf. [MS64]), so that $W^{k,p}(\Omega)$ may alternatively be defined as the completion of the space of smooth functions in $\Omega$ whose derivatives of order less than or equal to $k$ belong to $L^p(\Omega)$. Such functions need not extend to continuous functions on $\bar{\Omega}$ or be bounded on $\Omega$.

**1.4. Embedding theorems.** Can we estimate the $L^q(\mathbb{R}^n)$ norm of a smooth, compactly supported function in terms of the $L^p(\mathbb{R}^n)$-norm of its derivative?

1.4.1. $1 \leqslant p < n$. We will show that, given $1 \leqslant p < n$, this is possible for a unique value of $q$, called the Sobolev conjugate of $p$. We are looking for an estimate of the form

$$\|f\|_{L^q(\mathbb{R}^n)} \leqslant C\|Df\|_{L^p(\mathbb{R}^n)} \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n), \tag{1.1}$$

for some constant $C = C(p,q,n)$. For $\lambda > 0$, let $f_\lambda$ denote the rescaled function

$$f_\lambda(x) := f\left(\frac{x}{\lambda}\right).$$

Then, changing variables $x \mapsto \lambda x$ in the integrals that define the $L^p, L^q$ norms, with $1 \leqslant p, q < \infty$, and using the fact that

$$Df_\lambda = \frac{1}{\lambda}(Df)_\lambda$$

we find that

$$\left(\int_{\mathbb{R}^n} |Df_\lambda|^p \,\mathrm{d}x\right)^{1/p} = \lambda^{n/p-1}\left(\int_{\mathbb{R}^n} |Df|^p \,\mathrm{d}x\right)^{1/p},$$
$$\left(\int_{\mathbb{R}^n} |f_\lambda|^q \,\mathrm{d}x\right)^{1/q} = \lambda^{n/q}\left(\int_{\mathbb{R}^n} |f|^q \,\mathrm{d}x\right)^{1/q}.$$

These norms must scale according to the same exponent if we are to have an inequality of the desired form, otherwise we can violate the inequality by taking $\lambda \to 0$ or $\lambda \to \infty$. The equality of exponents implies that $q = p^*$ where $p^*$ satisfies

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

Note that we need $1 \leqslant p < n$ to ensure that $p^* > 0$, in which case $p < p^* < \infty$. We assume that $n \geqslant 2$. From this, we motivate the following definition.

Definition 4.4 (Sobolev conjugate exponents). *If* $1 \leqslant p < n$, *then the* Sobolev conjugate $p^*$ *of* $p$ *is*

$$p^* = \frac{np}{n-p}$$

Thus, an estimate of the form (1.1) is possible only if $q = p^*$; we will show that (1.1) is, in fact, true when $q = p^*$.

This result was obtained by Sobolev [Sob38] for $1 < p < n$ and by Emilio Gagliardo and Luis Nirenberg independently up to the endpoint $p = 1$ (see [Nir59; Gag58])

Theorem 4.2 (Gagliardo–Nirenberg–Sobolev's inequality). *Let* $1 \leqslant p < n$, *where* $n \geqslant 2$, *and let* $p^*$ *be the Sobolev conjugate of* $p$ *given in Definition 4.4. Then*

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \leqslant C\|Df\|_{L^p(\mathbb{R}^n)}, \qquad \textit{for all } f \in C_c^\infty(\mathbb{R}^n),$$

*where*

$$C(n,p) = \frac{p}{2n}\left(\frac{n-1}{n-p}\right). \tag{1.2}$$

Remark 4.3 (Optimal Sobolev constant). *The constant stated in Theorem 4.2 is not optimal. Instead, for* $p = 1$, *the best constant is*

$$C(n,1) = \frac{1}{n\alpha_n^{1/n}},$$

*where* $\alpha_n$ *is the volume of the unit ball, or*

$$C(n,1) = \frac{1}{n\sqrt{\pi}}\left[\Gamma\left(1+\frac{n}{2}\right)\right]^{1/n}$$

*where* $\Gamma$ *is the Euler Gamma function. Equality is obtained in the limit of functions that approach the characteristic function of a ball. This result for the best Sobolev constant is equivalent to the isoperimetric inequality that a sphere has minimal area among all surfaces enclosing a given volume.*

*For* $1 < p < n$, *the best constant is*

$$C(n,p) = \frac{1}{n^{1/p}\sqrt{\pi}}\left(\frac{p-1}{n-p}\right)^{1-1/p}\left[\frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)}\right]^{1/n}$$

*and equality holds for functions of the form*

$$f(x) = \left(a + b|x|^{p/(p-1)}\right)^{1-n/p},$$

*where* $a$, $b$ *are positive constants, which are called* Aubin–Talenti bubbles.[9]

Example 4.2. *The Sobolev inequality in Theorem 4.2 does not hold in the limiting case* $p \to n, p^* \to \infty$. *If* $\phi(x)$ *is a smooth cut-off function that is equal to one for* $|x| \leqslant 1$ *and zero for* $|x| \geqslant 2$, *and*

$$f(x) = \phi(x)\log\log\left(1+\frac{1}{|x|}\right),$$

*then* $Df \in L^n(\mathbb{R}^n)$, *and* $f \in W^{1,n}(\mathbb{R})$, *but* $f \notin L^\infty(\mathbb{R}^n)$.

Before describing the proof, we introduce some notation, explain the main idea, and establish a preliminary inequality.

For $1 \leqslant i \leqslant n$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, let

$$x_i' = (x_1, \ldots, \hat{x}_i, \ldots x_n) \in \mathbb{R}^{n-1}$$

where the 'hat' means that the $i$-th coordinate is omitted. We write $x = (x_i, x_i')$ and denote the value of a function $f : \mathbb{R}^n \to \mathbb{R}$ at $x$ by

$$f(x) = f\left(x_i, x_i'\right).$$

If $f$ is smooth with compact support, then the fundamental theorem of calculus implies that

$$f(x) = \int_{-\infty}^{x_i} \partial_{x_i} f\left(t, x_i'\right) \,\mathrm{d}t$$

[9] Named after Thierry Aubin [Aub76] and Giorgio Talenti [Tal76].

Taking absolute values, we get

$$|f(x)| \leqslant \int_{-\infty}^{\infty} \left|\partial_{x_i} f\left(t, x_i'\right)\right| \mathrm{d}t.$$

We can improve the constant in this estimate by using the fact that

$$\int_{-\infty}^{\infty} \partial_{x_i} f\left(t, x_i'\right) \mathrm{d}t = 0.$$

LEMMA 4.7. *Let us suppose that $g : \mathbb{R} \to \mathbb{R}$ is an integrable function with compact support such that $\int g \,\mathrm{d}t = 0$. If*

$$f(x) = \int_{-\infty}^{x} g(t) \,\mathrm{d}t,$$

*then*

$$|f(x)| \leqslant \frac{1}{2} \int |g(t)| \,\mathrm{d}t.$$

PROOF. Let $g = g_+ - g_-$ where the nonnegative functions $g_+, g_-$ are defined by $g_+ = \max(g, 0), g_- = \max(-g, 0)$. Then $|g| = g_+ + g_-$ and

$$\int g_+ dt = \int g_- dt = \frac{1}{2} \int |g| dt.$$

It follows that

$$f(x) \leqslant \int_{-\infty}^{x} g_+(t) \,\mathrm{d}t \leqslant \int_{-\infty}^{\infty} g_+(t) \,\mathrm{d}t = \frac{1}{2} \int |g| \,\mathrm{d}t$$
$$f(x) \geqslant -\int_{-\infty}^{x} g_-(t) \,\mathrm{d}t \geqslant -\int_{-\infty}^{\infty} g_-(t) dt = -\frac{1}{2} \int |g| \,\mathrm{d}t,$$

which proves the result. □

Thus, for $1 \leqslant i \leqslant n$ we have

$$|f(x)| \leqslant \frac{1}{2} \int_{-\infty}^{\infty} \left|\partial_{x_i} f\left(t, x_i'\right)\right| \mathrm{d}t.$$

The idea of the proof is to average a suitable power of this inequality over the $i$-directions and integrate the result to estimate $f$ in terms of $Df$. In order to do this, we use the following inequality, which estimates the $L^1$-norm of a function of $x \in \mathbb{R}^n$ in terms of the $L^{n-1}$-norms of $n$ functions of $x_i' \in \mathbb{R}^{n-1}$ whose product bounds the original function pointwise.

Thus, for $1 \leqslant i \leqslant n$ we have

$$|f(x)| \leqslant \frac{1}{2} \int_{-\infty}^{\infty} \left|\partial_{x_i} f\left(t, x_i'\right)\right| dt.$$

The idea of the proof is to average a suitable power of this inequality over the $i$-directions and integrate the result to estimate $f$ in terms of $Df$. In order to do this, we use the following inequality, which estimates the $L^1$-norm of a function of $x \in \mathbb{R}^n$ in terms of the $L^{n-1}$-norms of $n$ functions of $x_i' \in \mathbb{R}^{n-1}$ whose product bounds the original function pointwise.

LEMMA 4.8 (Gagliardo's product inequality). *Let us suppose that $n \geqslant 2$ and*

$$\left\{g_i \in C_c^{\infty}\left(\mathbb{R}^{n-1}\right) : 1 \leqslant i \leqslant n\right\}$$

*are non-negative functions. If we define $g \in C_c^{\infty}\left(\mathbb{R}^n\right)$ by*

$$g(x) := \prod_{i=1}^{n} g_i\left(x_i'\right),$$

*then*

$$\int g \,\mathrm{d}x \leqslant \prod_{i=1}^{n} \|g_i\|_{n-1} \tag{1.3}$$

REMARK 4.4. *If $n = 2$, Lemma 4.8 states that*

$$\int g_1(x_2)\, g_2(x_1)\, \mathrm{d}x_1\, \mathrm{d}x_2 \leqslant \left(\int g_1(x_2)\, \mathrm{d}x_2\right)\left(\int g_2(x_1)\, \mathrm{d}x_1\right),$$

*which follows immediately from Fubini's theorem.*

*If $n = 3$, Lemma 4.8 states that*

$$\int g_1(x_2, x_3)\, g_2(x_1, x_3)\, g_3(x_1, x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3$$
$$\leqslant \left(\int g_1^2(x_2, x_3)\, \mathrm{d}x_2\, \mathrm{d}x_3\right)^{1/2}\left(\int g_2^2(x_1, x_3)\, \mathrm{d}x_1\, \mathrm{d}x_3\right)^{1/2}\left(\int g_3^2(x_1, x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2\right)^{1/2}.$$

*To prove the inequality in this case, we fix $x_1$ and apply Cauchy–Schwartz' inequality to the $x_2x_3$-integral of $g_1 \cdot g_2g_3$. We then use the inequality for $n = 2$ to estimate the $x_2x_3$-integral of $g_2g_3$, and integrate the result over $x_1$. An analogous approach works for higher $n$.*

REMARK 4.5. *Note that under the scaling $g_i \mapsto \lambda g_i$, both sides of (1.3) scale in the same way,*

$$\int g\, \mathrm{d}x \mapsto \left(\prod_{i=1}^{n} \lambda_i\right)\int g\, \mathrm{d}x, \qquad \prod_{i=1}^{n} \|g_i\|_{n-1} \mapsto \left(\prod_{i=1}^{n} \lambda_i\right)\prod_{i=1}^{n} \|g_i\|_{n-1}$$

*as must be true for any inequality involving norms. Also, under the spatial rescaling $x \mapsto \lambda x$, we have*

$$\int g dx \mapsto \lambda^{-n}\int g dx$$

*while $\|g_i\|_p \mapsto \lambda^{-(n-1)/p}\|g_i\|_p$, so*

$$\prod_{i=1}^{n} \|g_i\|_p \mapsto \lambda^{-n(n-1)/p}\prod_{i=1}^{n} \|g_i\|_p$$

*Thus, if $p = n - 1$ the two terms scale in the same way, which explains the appearance of the $L^{n-1}$-norms of the $g_i$ 's on the right hand side of (1.3).*

PROOF. We argue by induction. We have the claim for $n = 2$ owing to Remark 4.5. Supposing that it is true for $n - 1$ where $n \geqslant 3$, we want to prove it for $n$.

For $1 \leqslant i \leqslant n$, let $g_i : \mathbb{R}^{n-1} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be the functions given in the theorem. Fix $x_1 \in \mathbb{R}$ and define $g_{x_1} : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$g_{x_1}(x_1') = g(x_1, x_1')$$

For $2 \leqslant i \leqslant n$, let $x_i' = (x_1, x_{1,i}')$ where

$$x_{1,i}' = (\hat{x}_1, \ldots, \hat{x}_i, \ldots x_n) \in \mathbb{R}^{n-2}$$

Define $g_{i,x_1} : \mathbb{R}^{n-2} \to \mathbb{R}$ and $\tilde{g}_{i,x_1} : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$g_{i,x_1}(x_{1,i}') = g_i(x_1, x_{1,i}').$$

Then

$$g_{x_1}(x_1') = g_1(x_1')\prod_{i=2}^{n} g_{i,x_1}(x_{1,i}').$$

Using Hölder's inequality with $q = n - 1$ and $q' = (n - 1)/(n - 2)$, we get

$$\int g_{x_1}\, \mathrm{d}x_1' = \int g_1\left(\prod_{i=2}^{n} g_{i,x_1}(x_{1,i}')\right) dx_1'$$
$$\leqslant \|g_1\|_{n-1}\left[\int\left(\prod_{i=2}^{n} g_{i,x_1}(x_{1,i}')\right)^{(n-1)/(n-2)} \mathrm{d}x_1'\right]^{(n-2)/(n-1)}$$

The induction hypothesis implies that

$$\begin{aligned}\int\left(\prod_{i=2}^{n} g_{i,x_1}\left(x'_{1,i}\right)\right)^{(n-1)/(n-2)} \mathrm{d}x'_1 &\leqslant \prod_{i=2}^{n}\left\|g_{i,x_1}^{(n-1)/(n-2)}\right\|_{n-2}\\ &\leqslant \prod_{i=2}^{n}\left\|g_{i,x_1}\right\|_{n-1}^{(n-1)/(n-2)}\end{aligned}$$

Hence,

$$\int g_{x_1}\,\mathrm{d}x'_1 \leqslant \|g_1\|_{n-1}\prod_{i=2}^{n}\|g_{i,x_1}\|_{n-1}$$

Integrating this equation over $x_1$ and using the generalized Hölder inequality with $p_2 = p_3 = \cdots = p_n = n-1$, we get

$$\begin{aligned}\int g dx &\leqslant \|g_1\|_{n-1}\int\left(\prod_{i=2}^{n}\|g_{i,x_1}\|_{n-1}\right)\mathrm{d}x_1\\ &\leqslant \|g_1\|_{n-1}\left(\prod_{i=2}^{n}\int\|g_{i,x_1}\|_{n-1}^{n-1}\,\mathrm{d}x_1\right)^{1/(n-1)}\end{aligned}$$

Thus, since

$$\begin{aligned}\int\|g_{i,x_1}\|_{n-1}^{n-1}\,\mathrm{d}x_1 &= \int\left(\int\left|g_{i,x_1}\left(x'_{1,i}\right)\right|^{n-1}\mathrm{d}x'_{1,i}\right)dx_1\\ &= \int\left|g_i\left(x'_i\right)\right|^{n-1}\mathrm{d}x'_i\\ &= \|g_i\|_{n-1}^{n-1}\end{aligned}$$

we find that

$$\int g\,\mathrm{d}x \leqslant \prod_{i=1}^{n}\|g_i\|_{n-1}$$

□

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. First, we prove the result for $p=1$ and then for $1<p<n$.
Case 1: $p=1$. For $1\leqslant i\leqslant n$, we have, by Lemma 4.7,

$$|f(x)| \leqslant \frac{1}{2}\int\left|\partial_{x_i}f\left(t,x'_i\right)\right|\mathrm{d}t$$

Multiplying these inequalities and taking the $(n-1)$-th root, we get

$$|f|^{n/(n-1)} \leqslant \frac{1}{2^{n/(n-1)}}g, \qquad \text{with } g := \prod_{i=1}^{n}\tilde{g}_i,$$

where $\tilde{g}_i(x) = g_i\left(x'_i\right)$ with

$$g_i\left(x'_i\right) = \left(\int\left|\partial_{x_i}f\left(t,x'_i\right)\right|\mathrm{d}t\right)^{1/(n-1)}.$$

Lemma 4.8 implies that

$$\int g\,\mathrm{d}x \leqslant \prod_{i=1}^{n}\|g_i\|_{n-1}$$

So, since

$$\|g_i\|_{n-1} = \left(\int\left|\partial_{x_i}f\right|\mathrm{d}x\right)^{1/(n-1)},$$

it follows that

$$\int|f|^{n/(n-1)}\,\mathrm{d}x \leqslant \frac{1}{2^{n/(n-1)}}\left(\prod_{i=1}^{n}\int\left|\partial_{x_i}f\right|\mathrm{d}x\right)^{1/(n-1)}.$$

Note that $n/(n-1) = 1^*$ is the Sobolev conjugate of 1.

Using the arithmetic–geometric mean inequality,

$$\left(\prod_{i=1}^{n} a_i\right)^{1/n} \leqslant \frac{1}{n}\sum_{i=1}^{n} a_i,$$

we get

$$\int |f|^{n/(n-1)}\, \mathrm{d}x \leqslant \left(\frac{1}{2n}\sum_{i=1}^{n}\int |\partial_{x_i} f|\, \mathrm{d}x\right)^{n/(n-1)},$$

i.e.,

$$\|f\|_{1^*} \leqslant \frac{1}{2n}\|Df\|_1.$$

<u>Case 2: $1 < p < n$</u> For any $s > 1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}|x|^s = s\, \mathrm{sign}\, x|x|^{s-1}.$$

Thus,

$$\begin{aligned} |f(x)|^s &= \int_{-\infty}^{x_i} \partial_{x_i}\left|f\left(t, x_i'\right)\right|^s\, \mathrm{d}t \\ &= s\int_{-\infty}^{x_i}\left|f\left(t, x_i'\right)\right|^{s-1}\mathrm{sgn}\left[f\left(t, x_i'\right)\right]\partial_{x_i} f\left(t, x_i'\right)\, \mathrm{d}t \end{aligned}$$

Using Lemma 4.7, it follows that

$$|f(x)|^s \leqslant \frac{s}{2}\int_{-\infty}^{\infty}\left|f^{s-1}\left(t, x_i'\right)\partial_{x_i} f\left(t, x_i'\right)\right|\, \mathrm{d}t$$

and multiplication of these inequalities gives

$$|f(x)|^{sn} \leqslant \left(\frac{s}{2}\right)^n\prod_{i=1}^{n}\int_{-\infty}^{\infty}\left|f^{s-1}\left(t, x_i'\right)\partial_{x_i} f\left(t, x_i'\right)\right|\, \mathrm{d}t.$$

Applying Lemma 4.8 with the functions

$$g_i\left(x_i'\right) = \left[\int_{-\infty}^{\infty}\left|f^{s-1}\left(t, x_i'\right)\partial_{x_i} f\left(t, x_i'\right)\right|\, \mathrm{d}t\right]^{1/(n-1)},$$

we obtain

$$\|f\|_{sn/(n-1)}^{sn} \leqslant \frac{s}{2}\prod_{i=1}^{n}\left\|f^{s-1}\partial_{x_i} f\right\|_1.$$

By Hölder's inequality, we have

$$\left\|f^{s-1}\partial_{x_i} f\right\|_1 \leqslant \left\|f^{s-1}\right\|_{p'}\left\|\partial_{x_i} f\right\|_p$$

On the other hand,

$$\left\|f^{s-1}\right\|_{p'} = \|f\|_{p'(s-1)}^{s-1}$$

Choosing $s > 1$ so that

$$p'(s-1) = \frac{sn}{n-1},$$

which holds if

$$s = p\left(\frac{n-1}{n-p}\right) \iff \frac{sn}{n-1} = p^*,$$

then

$$\|f\|_{p^*} \leqslant \frac{s}{2}\left(\prod_{i=1}^{n}\left\|\partial_{x_i} f\right\|_p\right)^{1/n}.$$

Using the arithmetic-geometric mean inequality, we get

$$\|f\|_{p^*} \leqslant \frac{s}{2n}\left(\sum_{i=1}^{n}\left\|\partial_{x_i} f\right\|_p^p\right)^{1/p},$$

which proves the result. □

We can use the Sobolev inequality to prove various embedding theorems.

DEFINITION 4.5. *We say that a Banach space $X$ is* continuously embedded *(or* embedded *for short) in a Banach space $Y$ if there is a one-to-one, bounded linear map $\imath : X \to Y$.*

We often think of $\iota$ as identifying elements of the smaller space $X$ with elements of the larger space $Y$; if $X$ is a subset of $Y$, then $\imath$ is the inclusion map. The boundedness of $\imath$ means that there is a constant $C$ such that $\|\imath x\|_Y \leqslant C\|x\|_X$ for all $x \in X$, so the weaker $Y$-norm of $\imath x$ is controlled by the stronger $X$-norm of $x$. We write an embedding as $X \hookrightarrow Y$ or as $X \subset Y$ when the boundedness is understood.

Theorem 4.2 does not, of course, imply that $f \in L^p(\mathbb{R}^n)$ whenever $Df \in L^p(\mathbb{R}^n)$, since constant functions have zero derivative.

To ensure that $f \in L^{p^*}(\mathbb{R}^n)$, we also need to impose a decay condition on $f$ that eliminates the constant functions. In the following theorem, this is provided by the assumption that $f \in L^p(\mathbb{R}^n)$

THEOREM 4.3. *Suppose that $1 \leqslant p < n$ and $p \leqslant q \leqslant p^*$ where $p^*$ is the Sobolev conjugate of $p$. Then $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ and $\|f\|_{L^q(\mathbb{R}^n)} \leqslant C\|f\|_{W^{1,p}(\mathbb{R}^n)}$ for all $f \in W^{1,p}(\mathbb{R}^n)$ for some constant $C = C(n,p,q)$.*

REMARK 4.6. *Let us present an heuristic argument for the lower bound $q \geqslant p$. Let us consider the function $f(x) = (1+|x|)^{-\alpha}$, with $\alpha > 0$. Then $f \in L^r(\mathbb{R}^n)$ if $\alpha r > n$. So, if $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, then $\alpha p > n$ should imply $\alpha q > n$ for every $\alpha > 0$, but this yields $p \leqslant q$ (otherwise there would exist $\alpha > 0$ sauch that $q < n/\alpha < p$.*

PROOF. Step 1. If $f \in W^{1,p}(\mathbb{R}^n)$, then, by Theorem 4.1, there is a sequence of functions $f_n \in C_c^\infty(\mathbb{R}^n)$ that converges to $f$ in $W^{1,p}(\mathbb{R}^n)$. Theorem 4.2 implies that $f_n \to f$ in $L^{p^*}(\mathbb{R}^n)$. In detail: $\{Df_n\}$ converges to $Df$ in $L^p$ so it is Cauchy in $L^p$; since

$$\|f_n - f_m\|_{p^*} \leqslant C\,\|Df_n - Df_m\|_p\,,$$

then $\{f_n\}$ is Cauchy in $L^{p^*}$; therefore $f_n \to \tilde f$ for some $\tilde f \in L^{p^*}$ since $L^{p^*}$ is complete; and $\tilde f$ is equivalent to $f$ since a subsequence of $\{f_n\}$ converges pointwise a.e. to $\tilde f$, because of the $L^{p^*}$ convergence, and to $f$, because of the $L^p$-convergence. Thus, $f \in L^{p^*}(\mathbb{R}^n)$ and

$$\|f\|_{p^*} \leqslant C\|Df\|_p$$

Step 2. Since $f \in L^p(\mathbb{R}^n)$, using the interpolation between $L^p$ spaces, we have, for $p < q < p^*$,

$$\|f\|_q \leqslant \|f\|_p^\theta \|f\|_{p^*}^{1-\theta}$$

where $0 < \theta < 1$ is defined by

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^*}.$$

Therefore, using Theorem 4.2 and the inequality

$$a^\theta b^{1-\theta} \leqslant \left[\theta^\theta(1-\theta)^{1-\theta}\right]^{1/p}(a^p+b^p)^{1/p}\,,$$

we get

$$\begin{aligned}\|f\|_q &\leqslant C^{1-\theta}\|f\|_p^\theta\|Df\|_p^{1-\theta}\\ &\leqslant C^{1-\theta}\left[\theta^\theta(1-\theta)^{1-\theta}\right]^{1/p}\left(\|f\|_p^p+\|Df\|_p^p\right)^{1/p}\\ &\leqslant C^{1-\theta}\left[\theta^\theta(1-\theta)^{1-\theta}\right]^{1/p}\|f\|_{W^{1,p}}.\end{aligned}$$

□

Instead of the assumption that $f \in L^p(\mathbb{R}^n)$, we can impose the following weaker decay condition.

DEFINITION 4.6. *A Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{R}$* vanishes at infinity *if for every $\epsilon > 0$ the set $\{x \in \mathbb{R}^n : |f(x)| > \epsilon\}$ has finite Lebesgue measure.*

The Sobolev embedding theorem remains true for functions that vanish at infinity.

THEOREM 4.4. *Suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is weakly differentiable with $Df \in L^p(\mathbb{R}^n)$ where $1 \leqslant p < n$ and $f$ vanishes at infinity. Then $f \in L^{p^*}(\mathbb{R}^n)$ and*

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \leqslant C\|Df\|_{L^p(\mathbb{R}^n)}$$

*where $C$ is given in (1.2).*

REMARK 4.7. *If $f \in L^p(\mathbb{R}^n)$ for some $1 \leqslant p < \infty$, then $f$ vanishes at infinity. Note that this does not imply that $\lim_{|x|\to\infty} f(x) = 0$.*

EXAMPLE 4.3. *Define $f : \mathbb{R} \to \mathbb{R}$ by*

$$f = \sum_{n\in\mathbb{N}} \chi_{I_n}, \quad I_n = \left[n, n + \frac{1}{n^2}\right]$$

*where $\chi_I$ is the characteristic function of the interval $I$. Then*

$$\int_{\mathbb{R}} f \,\mathrm{d}x = \sum_{n\in\mathbb{N}} \frac{1}{n^2} < \infty$$

*so $f \in L^1(\mathbb{R})$. The limit of $f(x)$ as $|x| \to \infty$ does not exist since $f(x)$ takes on the values 0 and 1 for arbitrarily large values of $x$. Nevertheless, $f$ vanishes at infinity since for any $\epsilon < 1$,*

$$|\{x \in \mathbb{R} : |f(x)| > \epsilon\}| = \sum_{n\in\mathbb{N}} \frac{1}{n^2},$$

*which is finite.*

EXAMPLE 4.4. *The function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} 1/\log x, & \text{if } x \geqslant 2, \\ 0, & \text{if } x < 2, \end{cases}$$

*vanishes at infinity, but $f \notin L^p(\mathbb{R})$ for any $1 \leqslant p < \infty$.*

1.4.2. *$p > n$.* If the weak derivative of a function that vanishes at infinity belongs to $L^p(\mathbb{R}^n)$ with $p < n$, then the function has improved integrability properties and belongs to $L^{p^*}(\mathbb{R}^n)$. Even though the function is weakly differentiable, it need not be continuous. In this section, we show that if the derivative belongs to $L^p(\mathbb{R}^n)$ with $p > n$ then the function (or a pointwise a.e. equivalent version of it) is continuous, and in fact Hölder continuous.

The following result is due to Charles Bradfield Morrey jun. (see [Mor40; Mor66]). The main idea is to estimate the difference $|f(x) - f(y)|$ in terms of $Df$ by the mean value theorem, average the result over a ball $B_r(x)$ and estimate the result in terms of $\|Df\|_p$ by Hölder's inequality.

THEOREM 4.5 (Morrey's inequality). *Let $n < p < \infty$ and*

$$\alpha = 1 - \frac{n}{p},$$

*with $\alpha = 1$ if $p = \infty$. Then there are constants $C = C(n,p)$ such that*

$$[f]_\alpha \leqslant C\|Df\|_p \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n) \tag{1.4}$$

$$\sup_{\mathbb{R}^n} |f| \leqslant C\|f\|_{W^{1,p}} \qquad \text{for all } f \in C_c^\infty(\mathbb{R}^n), \tag{1.5}$$

*where $[\cdot]_\alpha$ denotes the Hölder seminorm $[\cdot]_{\alpha,\mathbb{R}^n}$, i.e.,*

$$|f|_{\alpha,\mathbb{R}^n} := \sup_{\substack{x,y\in\mathbb{R}^n,\\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.$$

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PROOF. Step 1. First we prove that there exists a constant $C$, depending only on $n$, such that, for any ball $B_r(x)$,

$$\fint_{B_r(x)} |f(x) - f(y)| \,\mathrm{d}y \leqslant C \int_{B_r(x)} \frac{|Df(y)|}{|x - y|^{n-1}} \,\mathrm{d}y. \tag{1.6}$$

Let $w \in \partial B_1(0)$ be a unit vector. For $s > 0$,

$$f(x + sw) - f(x) = \int_0^s \frac{\mathrm{d}}{\mathrm{d}t} f(x + tw) \,\mathrm{d}t = \int_0^s Df(x + tw) \cdot w \,\mathrm{d}t,$$

and, therefore, since $|w| = 1$,

$$|f(x + sw) - f(x)| \leqslant \int_0^s |Df(x + tw)| \,\mathrm{d}t.$$

Integrating this inequality with respect to $w$ over the unit sphere, we get

$$\int_{\partial B_1(0)} |f(x) - f(x+sw)| \, \mathrm{d}S(w) \leqslant \int_{\partial B_1(0)} \left( \int_0^s |Df(x+tw)| \, \mathrm{d}t \right) \mathrm{d}S(w).$$

Changing the order of integration yields

$$\begin{aligned} \int_{\partial B_1(0)} \left( \int_0^s |Df(x+tw)| \, \mathrm{d}t \right) \mathrm{d}S(w) &= \int_{\partial B_1(0)} \int_0^s \frac{|Df(x+tw)|}{t^{n-1}} t^{n-1} \, \mathrm{d}t \, \mathrm{d}S(w) \\ &= \int_{B_x(x)} \frac{|Df(y)|}{|x-y|^{n-1}} \, \mathrm{d}y. \end{aligned}$$

Thus,

$$\int_{\partial B_1(0)} |f(x) - f(x+sw)| \, \mathrm{d}S(w) \leqslant \int_{B_x(x)} \frac{|Df(y)|}{|x-y|^{n-1}} \, \mathrm{d}y.$$

Using this inequality, we can also estimate

$$\begin{aligned} \int_{B_r(x)} |f(x) - f(y)| \, \mathrm{d}y &= \int_0^r \left( \int_{\partial B_1(0)} |f(x) - f(x+sw)| \, \mathrm{d}S(w) \right) s^{n-1} \, \mathrm{d}s \\ &\leqslant \int_0^r \left( \int_{B_s(x)} \frac{|Df(y)|}{|x-y|^{n-1}} \, \mathrm{d}y \right) s^{n-1} \, \mathrm{d}s \\ &\leqslant \left( \int_0^r s^{n-1} \, \mathrm{d}s \right) \left( \int_{B_{r(x)}} \frac{|Df(y)|}{|x-y|^{n-1}} \, \mathrm{d}y \right) \\ &\leqslant \frac{r^n}{n} \int_{B_r(x)} \frac{|Df(y)|}{|x-y|^{n-1}} \, \mathrm{d}y, \end{aligned}$$

where we estimated the integral over $B_s(x)$ by the integral over $B_r(x)$ for $s \leqslant r$. This gives (1.6) with $C = (n\alpha_n)^{-1}$.

Step 2. To prove (1.4), let us suppose that $x, y \in \mathbb{R}^n$. Let $r = |x-y|$ and $\Omega = B_r(x) \cap B_r(y)$. Then averaging the inequality

$$|f(x) - f(y)| \leqslant |f(x) - f(z)| + |f(y) - f(z)|$$

with respect to $z$ over $\Omega$, we get

$$|f(x) - f(y)| \leqslant \fint_\Omega |f(x) - f(z)| \, \mathrm{d}z + \fint_\Omega |f(y) - f(z)| \, \mathrm{d}z. \tag{1.7}$$

From (1.6) and Hölder's inequality,

$$\begin{aligned} \fint_\Omega |f(x) - f(z)| \, \mathrm{d}z &\leqslant \fint_{B_r(x)} |f(x) - f(z)| \, \mathrm{d}z \\ &\leqslant C \int_{B_r(x)} \frac{|Df(y)|}{|x-y|^{n-1}} \, \mathrm{d}y \\ &\leqslant C \left( \int_{B_{r(x)}} |Df|^p \, \mathrm{d}z \right)^{1/p} \left( \int_{B_{r(x)}} \frac{\mathrm{d}z}{|x-z|^{p'(n-1)}} \right)^{1/p'} \end{aligned}$$

We have

$$\left( \int_{B_r(x)} \frac{\mathrm{d}z}{|x-z|^{p'(n-1)}} \right)^{1/p'} = C \left( \int_0^r \frac{r^{n-1} dr}{r^{p'(n-1)}} \right)^{1/p'} = Cr^{1-n/p},$$

where $C$ denotes a generic constant depending on $n$ and $p$. Thus,

$$\fint_\Omega |f(x) - f(z)| \, \mathrm{d}z \leqslant Cr^{1-n/p} \|Df\|_{L^p(\mathbb{R}^n)}$$

with a similar estimate for the integral in which $x$ is replaced by $y$. Using these estimates in (1.7) and setting $r = |x-y|$, we get

$$|f(x) - f(y)| \leqslant C|x-y|^{1-n/p} \|Df\|_{L^p(\mathbb{R}^n)}, \tag{1.8}$$

which proves (1.4)

<u>Step 3.</u> Finally, we prove (1.5). For any $x \in \mathbb{R}^n$, using (1.8), we find that

$$\begin{aligned}|f(x)| &\leqslant \fint_{B_1(x)} |f(x) - f(y)|\,\mathrm{d}y + \fint_{B_1(x)} |f(y)|\,\mathrm{d}y \\ &\leqslant C\|Df\|_{L^p(\mathbb{R}^n)} + C\|f\|_{L^p(B_1(x))} \\ &\leqslant C\|f\|_{W^{1,p}(\mathbb{R}^n)}\end{aligned}$$

and taking the supremum with respect to $x$, we get (1.5). □

Combining these estimates for

$$\|f\|_{C^{0,\alpha}} = \sup|f| + [f]_\alpha$$

and using a density argument, we can prove the following theorem. We denote by $C_0^{0,\alpha}(\mathbb{R}^n)$ the space of Hölder continuous functions $f$ whose limit as $x \to \infty$ is zero, meaning that for every $\epsilon > 0$ there exists a compact set $K \subset \mathbb{R}^n$ such that $|f(x)| < \epsilon$ if $x \in \mathbb{R}^n \backslash K$.

THEOREM 4.6. *Let $n < p < \infty$ and $\alpha = 1 - n/p$. Then*

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow C_0^{0,\alpha}(\mathbb{R}^n)$$

*and there exists a constant $C = C(n,p)$ such that*

$$\|f\|_{C^{0,\alpha}} \leqslant C\|f\|_{W^{1,p}} \qquad \textit{for all } f \in C_c^\infty(\mathbb{R}^n)$$

For $p = \infty$, we have that $f \in W^{1,\infty}(\mathbb{R}^n)$ is globally Lipschitz continuous, with

$$[f]_1 \leqslant C\|Df\|_{L^\infty(\mathbb{R}^n)}$$

A function in $W^{1,\infty}(\mathbb{R}^n)$ need not approach zero at infinity. We have in this case the following characterization of Lipschitz functions.

THEOREM 4.7. *A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is globally Lipschitz continuous if and only if it is weakly differentiable and $Df \in L^\infty(\mathbb{R}^n)$.*

When $n < p \leqslant \infty$, the above estimates can be used to prove that the pointwise derivative of a Sobolev function exists almost everywhere and agrees with the weak derivative.

THEOREM 4.8. *If $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $n < p \leqslant \infty$, then $f$ is differentiable pointwise a.e. and the pointwise derivative coincides with the weak derivative.*

1.4.3. *General embedding theorem.* More generally, we state the following result.

THEOREM 4.9. *Let $m \in \mathbb{N}^*$, $1 \leqslant p \leqslant \infty$. We have:*

- *If $kp < n$: $W^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, for $p \leqslant q \leqslant \frac{np}{n-kp}$;*
- *If $kp = n$: $W^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, for $p \leqslant q < \infty$;*
- *If $kp > n$: $W^{k,p}(\mathbb{R}^n) \hookrightarrow C^{m,\alpha}(\mathbb{R}^n)$, with* $\begin{cases} m = \left\lfloor k - \frac{n}{p} \right\rfloor, \ \alpha = \left\{k - \frac{n}{p}\right\}, & \textit{if } \frac{n}{p} \notin \mathbb{N}, \\ m = k - \frac{n}{p} - 1, \ \alpha = 1, & \textit{if } \frac{n}{p} \in \mathbb{N}, \end{cases}$

  *where $\lfloor \cdot \rfloor$ denotes the* integer part function *and $\{\cdot\}$ the* fractional part function.

EXAMPLE 4.5. *There exists a function $u \in H^1(\mathbb{R}^2)$ but $u \notin L^\infty(\mathbb{R}^2)$. Let $\psi \in \mathcal{D}(\mathbb{R}^2)$ such that $0 \leqslant \psi \leqslant 1$ and*

$$\psi(x) = \begin{cases} 1 & \textit{if } |x| < \frac{1}{2}, \\ 0 & \textit{if } |x| > \frac{3}{4}. \end{cases}$$

*The desired counterexample is given by*

$$u(x) = |\ln|x||^\alpha \psi(x), \quad 0 < \alpha < \frac{1}{2}.$$

*Since there is a logarithmic pole at $x = 0$, the function is not in $L^\infty(\mathbb{R}^2)$. We show that $u \in H^1(\mathbb{R}^2)$. It follows that $u \in L^2(\mathbb{R}^2)$, since*

$$\int_{\mathbb{R}^2} |u(x)|^2\,\mathrm{d}x \leqslant \int_{|x| \leqslant \frac{3}{4}} |\ln|x||^{2\alpha}\,\mathrm{d}x = 2\pi\int_0^{\frac{3}{4}} |\ln\rho|^{2\alpha}\rho\,\mathrm{d}\rho.$$

*There are no issues with the integrability of the given function. Observe that $u$ is $\mathcal{C}^\infty$ for $x \neq 0$. Therefore, for $x \neq 0$, the function admits classical derivatives. We show that these derivatives belong to $L^2$.*

$$\int_{\mathbb{R}^2} |D_j u(x)|^2 \,\mathrm{d}x \leqslant \int_{\mathbb{R}^2} \left(\alpha |\ln|x||^{\alpha-2} \ln|x| \frac{1}{|x|}\frac{x_j}{|x|}\psi + |\ln|x||^\alpha \partial_{x_j}\psi(x)\right)^2 \mathrm{d}x$$
$$\leqslant 2\alpha^2 \int_{|x|\leqslant \frac{3}{4}} |\ln|x||^{2(\alpha-1)} \frac{1}{|x|^2}\frac{x_j^2}{|x|^2}\,\mathrm{d}x + 2\|\nabla\psi\|_\infty^2 \int_{\frac{1}{2}<|x|<\frac{3}{4}} |\ln|x||^{2\alpha}\,\mathrm{d}x.$$

*The second integral is evidently finite. For the first integral, we switch to polar coordinates:*

$$\int_{|x|\leqslant\frac{3}{4}} \frac{|\ln|x||^{2(\alpha-1)}}{|x|^2}\,\mathrm{d}x = 2\pi \int_0^{\frac{3}{4}} \frac{|\ln\rho|^{2(\alpha-1)}}{\rho}\,\mathrm{d}\rho = 2\pi\left[\frac{(\ln\rho)^{2\alpha-1}}{2\alpha-1}\right]_0^{\frac{3}{4}}.$$

*Since $\alpha < \frac{1}{2}$, the integral converges as $\rho \to 0$. To conclude that $u \in H^1(\mathbb{R}^2)$, we must show that these classical derivatives coincide with the distributional derivatives, which is non-trivial because $x = 0$ is a singular point of the derivative. Let $\phi \in \mathcal{D}(\mathbb{R}^2)$. We want to prove that*

$$\int_{\mathbb{R}^2} D_j u\,\phi\,\mathrm{d}x = -\int_{\mathbb{R}^2} u\,\frac{\partial\phi}{\partial x_j}\,\mathrm{d}x.$$

*Starting from the second term:*

$$\int_{\mathbb{R}^2} u\,\frac{\partial\phi}{\partial x_j}\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^2\setminus B_\varepsilon(0)} u\,\frac{\partial\phi}{\partial x_j}\,\mathrm{d}x.$$

*This identity holds as a consequence of the dominated convergence theorem. Outside the origin, we can integrate by parts using classical derivatives:*

$$\int_{\mathbb{R}^2} u\,\frac{\partial\phi}{\partial x_j}\,\mathrm{d}x = \lim_{\varepsilon\to 0}\left(-\int_{\mathbb{R}^2\setminus B_\varepsilon(0)} \frac{\partial u}{\partial x_j}\phi\,\mathrm{d}x + \int_{\partial B_\varepsilon(0)} u\phi\nu_j\,\mathrm{d}\sigma\right).$$

*Here, $\nu_j$ is the $j$-th component of the outward normal vector. Using Lebesgue's theorem again for the first term, we have*

$$\int_{\mathbb{R}^2} u\,\frac{\partial\phi}{\partial x_j}\,\mathrm{d}x = -\int_{\mathbb{R}^2} \frac{\partial u}{\partial x_j}\phi\,\mathrm{d}x + \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(0)} u\phi\nu_j\,\mathrm{d}\sigma.$$

*To conclude, we must show that*

$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(0)} u\phi\nu_j\,\mathrm{d}\sigma = 0.$$

*For $\varepsilon < 1$, we have $u|_{B_\varepsilon(0)} = |\ln\varepsilon|^\alpha\psi(\varepsilon) = |\ln\varepsilon|^\alpha$. Thus,*

$$\int_{\partial B_\varepsilon(0)} u\phi\nu_j\,\mathrm{d}\sigma = |\ln\varepsilon|^\alpha \left|\int_{\partial B_\varepsilon(0)} \phi\nu_j\,\mathrm{d}\sigma\right| \leqslant |\ln\varepsilon|^\alpha \|\phi\|_\infty 2\pi\varepsilon.$$

Example 4.6. *There exists a function in $H^1(\mathbb{R}^2)$ but not in $L^\infty_{loc}(\mathbb{R}^2)$, meaning it is unbounded in any open subset of $\mathbb{R}^2$, not merely at a single point as in the previous proof.*

*Let $u$ be a positive function in $H^1(\mathbb{R}^2)$ such that $\lim_{x\to 0} u(x) = +\infty$. For instance, one can choose the function defined in the previous proof. Let $\{x_n\}$ be a dense sequence in $\mathbb{R}^2$, and consider the series*

$$\sum_{k=0}^{+\infty} 2^{-k} u(x - x_k).$$

*We show that this series converges in $H^1(\mathbb{R}^2)$. Indeed, since the $H^1(\mathbb{R}^2)$-norm is invariant under translations, we have*

$$\sum_{k=0}^{+\infty} 2^{-k}\|u(\cdot - x_k)\|_{H^1(\mathbb{R}^2)} = \sum_{k=0}^{+\infty} 2^{-k}\|u\|_{H^1(\mathbb{R}^2)} = 2\|u\|_{H^1(\mathbb{R}^2)}.$$

*Since $H^1(\mathbb{R}^2)$ is a Banach space, any normally convergent series also converges. Denote by $w$ the sum of the series in the sense of $H^1(\mathbb{R}^2)$.*

*On the other hand, by the construction of u, the series has positive terms, so at each point, the series either converges or diverges positively. Observe that w is also the pointwise limit. Indeed, convergence in* $H^1(\mathbb{R}^2)$ *to w implies convergence in* $L^2(\mathbb{R}^2)$ *to w, which in turn implies the pointwise convergence almost everywhere of a subsequence of the partial sums to w.*

*Fix* $h \in \mathbb{N}$*, and consider* $\lim_{x \to x_h} u(x - x_h) = +\infty$*. Therefore, at every point* $x_h$*, we have* $\lim_{x \to x_h} w(x) = +\infty$*.*

*However, since the given sequence is dense in* $\mathbb{R}^2$*, the function* $w \in H^1(\mathbb{R}^2)$*, but it is not bounded on any open subset of* $\mathbb{R}^2$*.*

**1.5. Boundary values of Sobolev functions.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $k \in \mathbb{N}$ and $1 \leqslant p < \infty$. After noticing that $\mathcal{D}(\Omega) \subset W^{k,p}(\Omega)$, we can define

$$W_0^{k,p}(\Omega) := \overline{\mathcal{D}}^{\|\cdot\|_{k,p}}$$

(the closure of $\mathcal{D}(\Omega)$ with respect to the norm $\|\cdot\|_{k,p}$). We also use the notation $H_0^k(\Omega) := W_0^{k,2}(\Omega)$.

PROPOSITION 4.1. *Let* $\Omega$ *be an open set with* $C^1$ *boundary and* $1 \leqslant p < \infty$*. If* $f \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$*, then*

$$u|_{\partial\Omega} = 0 \iff u \in W_0^{1,p}(\Omega).$$

REMARK 4.8. *To generalize this result to the case* $k \geqslant 1$*, we need to ask* $\Omega$ *to be of class* $C^k$ *and one proves that all the derivatives up to* $m-1$ *order are null at the boundary. In other words, one should not confuse* $W_0^{1,p} \cap W^{k,p}$ *with* $W_0^{k,p}$*.*

We have already discussed that, if $\Omega = \mathbb{R}^n$, then $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$. On the other hand, the following results hold.

THEOREM 4.10 (Meyers–Serrin's theorem). *Assume* $\Omega$ *is bounded and let* $u \in W^{k,p}(\Omega)$*, with* $1 \leqslant p < \infty$*. Then there exist functions* $u_m \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ *such that*

$$u_m \to u \qquad \text{in } W^{k,p}(\Omega) \text{ as } m \to \infty.$$

*In other words,* $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ *is dense in* $W^{k,p}(\Omega)$*.*

THEOREM 4.11. *Assume* $\Omega$ *is bounded and* $\partial\Omega \in C^1$*. Let* $u \in W^{k,p}(\Omega)$*, with* $1 \leqslant p < \infty$*. Then there exist functions* $u_m \in C^\infty(\bar{\Omega})$ *such that*

$$u_m \to u \qquad \text{in } W^{k,p}(\Omega) \text{ as } m \to \infty.$$

*In other words,* $C^\infty(\bar{\Omega})$ *is dense in* $W^{k,p}(\Omega)$*.*

1.5.1. *Extension operator.* We consider now the question whether, given a function $f \in W^{k,p}(\Omega)$, it is possible to construct an extension $\tilde{f} \in W^{k,p}(\mathbb{R}^n)$ such that $\tilde{f}\big|_\Omega = f$ and $\|\tilde{f}\|_{W^{k,p}(\mathbb{R}^n)} \leqslant C\|f\|_{W^{k,p}(\Omega)}$ for some $C > 0$ independent of $f$.

For $f \in L^p(\Omega)$, we can take the simple extension by zero outside the domain: $\tilde{f} = f$ in $\Omega$ and $\tilde{f} = 0$ in $\Omega^c$. Such extension clearly satisfies $\tilde{f} \in L^p(\mathbb{R}^n)$ and $\|\tilde{f}\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\Omega)}$ for any $p \in [1, \infty]$.

When it comes to functions in $W^{k,p}(\Omega)$, $k \geqslant 1$, however, the extension by zero outside $\Omega$ does not lead, in general, to a function in $W^{k,p}(\mathbb{R}^n)$ so the procedure to construct an extension operator is more delicate.

DEFINITION 4.7 (Extension operator). *We say that* $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$ *is an* extension operator *if*

1. *E is linear, i.e.,* $E(\alpha f + \beta g) = \alpha Ef + \beta Eg$ *for all* $f, g \in W^{k,p}(\Omega)$ *and for all* $\alpha, \beta \in \mathbb{R}$*;*
2. *E is bounded, i.e., there exists* $C > 0$ *such that* $\|Ef\|_{W^{k,p}(\mathbb{R}^n)} \leqslant C\|f\|_{W^{k,p}(\Omega)}$ *for all* $f \in W^{k,p}(\Omega)$*;*
3. $Ef = f$ *a.e. in* $\Omega$ *for all* $f \in W^{k,p}(\Omega)$*;*
4. *if* $\Omega$ *is bounded,* $E(f)$ *is compactly supported for all* $f \in W^{k,p}(\Omega)$*.*

THEOREM 4.12 (Extension theorem). *Let* $\Omega$ *be a domain in* $\mathbb{R}^n$ *with bounded boundary* $\partial\Omega$ *of class* $C^k$*. Then there exists an extension operator* $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$ *for any* $1 \leqslant p < \infty$*.*

1.5.2. *Trace theorems.* Let $\Omega \in \mathbb{R}^n$ be a domain with nonempty bounded and smooth boundary $\partial\Omega$, say of class $C^1$. We recall that a function $f \in L^p(\Omega)$ is defined only up to zero measure sets (i.e., it is defined only almost everywhere). Since $|\partial\Omega| = 0$, it is meaningless to talk about the "boundary value" of $f$ since we can modify the value of $f$ on $\partial\Omega$ without changing the equivalence class to which $f$ belongs. We ask now whether for a function $f \in W^{k,p}(\Omega)$, with $k \geqslant 1$, the picture changes, i.e., whether it makes sense to talk about the value of $f$ (also called the *trace*) on $\partial\Omega$.

It turns out that the answer is yes. The way to proceed is the following: since $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$, given $f \in W^{k,p}(\Omega)$ we can find a sequence $f_k \in C^\infty(\bar{\Omega})$ such that $f_k \xrightarrow{k\to\infty} f$ in $W^{k,p}(\Omega)$. For each $f_k$, the trace $f_k|_{\partial\Omega}$ is uniquely defined, so that we define the *trace of $f$ on* $\partial\Omega$ as $\lim_{k\to\infty} f_k|_{\partial\Omega}$. The crucial question is whether such limit exists and with respect to which topology. The following theorem answers this question.

THEOREM 4.13 (Trace theorem). *Let $\Omega \subset \mathbb{R}^n$ be a domain with bounded boundary $\partial\Omega$ of class $C^1$. Then there exists a linear and bounded operator $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, $1 \leqslant p < \infty$ such that $\tau f = f|_{\partial\Omega}$ for any $f \in C^0(\bar{\Omega}) \cap W^{1,p}(\Omega)$.*

From this, we may deduce the following result.

THEOREM 4.14 (Trace-zero functions in $W^{1,p}$). *Let us assume that $\Omega$ is bounded and $\partial\Omega$ is $C^1$. let us suppose, furthermore, that $u \in W^{1,p}(\Omega)$. Then $u \in W_0^{1,p}(\Omega)$ if and only if $\tau u = 0$ on $\partial\Omega$.*

**1.6. Compactness results.** A Banach space $X$ is *compactly embedded* in a Banach space $Y$, written $X \hookrightarrow\hookrightarrow Y$ or $X \Subset Y$, if the embedding $\imath : X \to Y$ is compact. That is, $\imath$ maps bounded sets in $X$ to precompact sets in $Y$; or, equivalently, if $\{x_n\}$ is a bounded sequence in $X$, then $\{\imath x_n\}$ has a convergent subsequence in $Y$.

An important property of the Sobolev embeddings is that they are compact on domains with finite measure. This corresponds to the rough principle that uniform bounds on higher derivatives imply compactness with respect to lower derivatives. The compactness of the Sobolev embeddings, due to Franz Rellich [Rel30] and Vladimir Iosifovich Kondrashov [Kon45].

THEOREM 4.15 (Rellich–Kondrashov's compact embedding theorem). *Suppose that $\Omega$ is a bounded open set in $\mathbb{R}^n$ with $C^1$ boundary, $k, m \in \mathbb{N}$ with $k \geqslant m$, and $1 \leqslant p < \infty$.*

*(1) If $kp < n$, then*

$$\begin{aligned} &W^{k,p}(\Omega) \Subset L^q(\Omega) \quad &&\text{for } 1 \leqslant q < np/(n-kp); \\ &W^{k,p}(\Omega) \subset L^q(\Omega) \quad &&\text{for } q = np/(n-kp) \end{aligned}$$

*More generally, if $(k-m)p < n$, then*

$$\begin{aligned} &W^{k,p}(\Omega) \Subset W^{m,q}(\Omega) \quad &&\text{for } 1 \leqslant q < np/(n-(k-m)p) \\ &W^{k,p}(\Omega) \subset W^{m,q}(\Omega) \quad &&\text{for } q = np/(n-(k-m)p) \end{aligned}$$

*(2) If $kp = n$, then*

$$W^{k,p}(\Omega) \Subset L^q(\Omega) \quad \text{for } 1 \leqslant q < \infty$$

*(3) If $kp > n$, then*

$$W^{k,p}(\Omega) \Subset C^{0,\mu}(\bar{\Omega})$$

*for $0 < \mu < k - n/p$ if $k - n/p < 1$, for $0 < \mu < 1$ if $k - n/p = 1$, and for $\mu = 1$ if $k - n/p > 1$; and*

$$W^{k,p}(\Omega) \subset C^{0,\mu}(\bar{\Omega})$$

*for $\mu = k - n/p$ if $k - n/p < 1$.*

*More generally, if $(k-m)p > n$, then*

$$W^{k,p}(\Omega) \Subset C^{m,\mu}(\bar{\Omega})$$

*for $0 < \mu < k - m - n/p$ if $k - m - n/p < 1$, for $0 < \mu < 1$ if $k - m - n/p = 1$, and for $\mu = 1$ if $k - m - n/p > 1$; and*

$$W^{k,p}(\Omega) \subset C^{m,\mu}(\bar{\Omega})$$

*for $\mu = k - m - n/p$ if $k - m - n/p = 0$. These results hold for arbitrary bounded open sets $\Omega$ if $W^{k,p}(\Omega)$ is replaced by $W_0^{k,p}(\Omega)$.*

Example 4.7. *If $u \in W^{n,1}(\mathbb{R}^n)$, then $u \in C_0(\mathbb{R}^n)$. This can be seen from the equality*

$$u(x) = \int_0^{x_1} \dots \int_0^{x_n} \partial_1 \cdots \partial_n u\left(x'\right) \mathrm{d}x_1' \dots \mathrm{d}x_n'$$

*which holds for all $u \in C_c^\infty(\mathbb{R}^n)$ and a density argument. In general, however, it is not true that $u \in L^\infty$ in the critical case $kp = n$.*

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**1.7. Poincaré's inequalities.** In some cases, the norm of $W_0^{1,p}(\Omega)$ may be simplified using Poincaré inequality [Poi90], which allows one to obtain bounds on a function using bounds on its derivatives and the geometry of its domain of definition.

Theorem 4.16 (Poincaré's inequality). *Suppose that $\Omega$ is an open subset of $\mathbb{R}^N$ bounded in one direction.*[10] *Then there is a constant $C_{Poi} > 0$ (depending only on $p$ and $\Omega$) such that, for any $u \in W_0^{1,p}(\Omega)$, with $p \in [1, \infty)$,*

$$\|u\|_{L^p(\Omega)} \leqslant C_{Poi} \|\nabla u\|_{L^p(\Omega)}.$$

Remark 4.9. *Finding the optimal constant in Poincaré's inequality, sometimes called the* Poincaré constant for the domain $\Omega$, *is, in general, a difficult that depends on the value of $p$ and the geometry of the domain $\Omega$. For example (and without pretense of completeness), we refer to [AD04; PW60] for the cases $p = 1$ and $p = 2$ in bounded, convex, Lipschitz domains.*

Proof. Without loss of generality, we write $x = (x_1, x')$ with $x' = (x_2, \dots, x_n)$ and suppose that $\Omega$ is bounded in the direction of $x_1$, i.e., there exists $a$, $b > 0$ such that, for all $x \in \Omega$, we have $x_1 \in ]a, b[$.

We assume that $u \in C_c^\infty(\Omega)$. This will be sufficient since, for any $u \in W_0^{1,p}(\Omega)$, there exists a sequence $\{u_k\}_k \subset C_0^\infty(\Omega)$ such that $u_k \xrightarrow{k \to \infty} 0$ in $W^{1,p}(\Omega)$. Therefore, if we prove Poincaré's inequality for $C_c^\infty$ functions, we can then pass to the limit and recover the statement for $W_0^{1,p}$ functions. We extend $u$ to the whole space $\mathbb{R}^n$ by setting $u(x) = 0$ for $x \notin \Omega$.

Step 1. We start with the case $p = 2$ (just because it is somewhat simpler). Using the variables $x = (x_1, x')$ with $x' = (x_2, \dots, x_n)$, we compute

$$u^2\left(x_1, x'\right) = \int_a^{x_1} 2u \partial_{x_1} u\left(t, x'\right) \mathrm{d}t.$$

An integration by parts yields

$$\begin{aligned} \|u\|_{L^2}^2 &= \int_{\mathbb{R}^n} u^2(x)\, \mathrm{d}x = \int_{\mathbb{R}^{n-1}} \int_a^b 1 \cdot \left( \int_a^{x_1} 2u \partial_{x_1} u\left(t, x'\right) \mathrm{d}t \right) \mathrm{d}x_1\, \mathrm{d}x' \\ &= \int_{\mathbb{R}^{n-1}} \int_a^b (b - x_1)\, 2u \partial_{x_1} u\left(x_1, x'\right) \mathrm{d}x_1\, \mathrm{d}x' \leqslant 2(b-a) \int_{\mathbb{R}^n} |u|\, |\partial_{x_1} u|\, \mathrm{d}x \\ &\leqslant 2(b-a) \|u\|_{L^2} \left\|\partial_{x_1} u\right\|_{L^2}. \end{aligned}$$

Dividing both sides by $\|u\|_{L^2}$, we obtain the result with $C_{\mathrm{Poi}} := 2(b-a)$.

Step 2. For $p \in [1, \infty)$, with $p \neq 2$, we the argument is similar and we leave it as an exercise. ☐

A generalization of this inequality to $W_0^{k,p}$ is available, due to Kurt Otto Friederichs [Fri27].

In particular, Poincaré's inequality holds true if $\Omega$ is supposed to be bounded. This result has the following important consequence.

Corollary 4.1. *If $\Omega$ is bounded in one direction or has a finite measure, then in $W_0^{1,p}(\Omega)$ the norm $\|f\|_{W_0^{1,p}(\Omega)} := \|Df\|_{L^p(\Omega)}$ is equivalent to the norm $\|f\|_{W^{1,p}(\Omega)}$, i.e., there exist two constants $C_1, C_2 > 0$ such that*

$$C_1 \|f\|_{W_0^{1,p}(\Omega)} \leqslant \|f\|_{W^{1,p}(\Omega)} \leqslant C_2 \|f\|_{W_0^{1,p}(\Omega)}.$$

[10] An open set $\Omega \subset \mathbb{R}^N$ is said to be *bounded in one direction* if there is $\mathrm{e} \in \mathbb{R}^N$, $\|\mathrm{e}\| = 1$, and two real numbers $a, b \in \mathbb{R}$ such that

$$x \cdot \mathrm{e} \in ]a, b[, \qquad \text{for any } x \in \Omega.$$

Proof. For every $f \in W_0^{1,p}(\Omega)$, we have $\|f\|^p_{L^p(\Omega)} \leqslant C^p_{\mathrm{Poi}} \|Df\|^p_{L^p(\Omega)}$ which implies

$$\|f\|^p_{W^{1,p}(\Omega)} = \|f\|^p_{L^p(\Omega)} + \|Df\|^p_{L^p(\Omega)} \leqslant (1 + C^p_{\mathrm{Poi}}) \|f\|^p_{W_0^{1,p}(\Omega)}$$

and this proves the second inequality with $C_2 = (1 + C^p_{\mathrm{Poi}})^{1/p}$. The first inequality trivially holds with $C_1 = 1$. □

In particular, Poincaré's inequality also implies that we may use as an equivalent inner-product on $H_0^1$ an expression that involves only the derivatives of the functions and not the functions themselves.

Corollary 4.2. *If $\Omega$ is an open set that is bounded in some direction, then $H_0^1(\Omega)$ equipped with the inner product*

$$(u, v)_0 = \int_\Omega Du \cdot Dv \,\mathrm{d}x, \tag{1.9}$$

*is a Hilbert space, and the corresponding norm is equivalent to the standard norm on $H_0^1(\Omega)$.*

Proof. We denote the norm associated with the inner-product (1.9) by

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |Du|^2 dx\right)^{1/2}$$

and the standard norm and inner product by

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega \left[u^2 + |Du|^2\right] \mathrm{d}x\right)^{1/2}$$

$$(u, v)_1 = \int_\Omega (uv + Du \cdot Dv) \,\mathrm{d}x.$$

Then, using Poincaré's inequality, we have

$$\|u\|_{H_0^1(\Omega)} \leqslant \|u\|_{H^1(\Omega)} \leqslant (C+1)^{1/2} \|u\|_0.$$

Thus, the two norms are equivalent; in particular, $\left(H_0^1, (\cdot,\cdot)_0\right)$ is complete since $\left(H_0^1, (\cdot,\cdot)_1\right)$ is complete, so it is a Hilbert space with respect to the inner product (1.9). □

For functions that do not vanish at the boundary, we can still prove an inequality of this type, attributed to Henri Poincaré and Wilhelm Wirtinger: as an application of the compact embedding theorem, we can prove an estimate on the difference between a function $u$ and its average value on a domain $\Omega$.

Theorem 4.17 (Poincaré–Wirtinger's inequality). *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, open set with $C^1$ boundary, and let $p \in [1, \infty]$. Then there exists a constant $C > 0$ (depending only on $p$ and $\Omega$), such that*

$$\left\| u - ⨍_\Omega u \,\mathrm{d}x \right\|_{L^p(\Omega)} \leqslant C \|\nabla u\|_{L^p(\Omega)} \qquad \text{for every } u \in W^{1,p}(\Omega).$$

Proof. Let us suppose, for the sake of finding a contradiction, that the conclusion is false. Then we could find a sequence of functions $u_k \in W^{1,p}(\Omega)$ with

$$\left\| u_k - ⨍_\Omega u_k \,\mathrm{d}x \right\|_{L^p(\Omega)} > k \|\nabla u_k\|_{L^p(\Omega)} \qquad \text{for every } k \in \{1, 2, \ldots\}.$$

Then the renormalized functions

$$v_k := \frac{u_k - ⨍_\Omega u_k \,\mathrm{d}x}{\left\| u_k - ⨍_\Omega u_k \,\mathrm{d}x \right\|_{L^p(\Omega)}}$$

satisfy

$$⨍_\Omega v_k \,\mathrm{d}x = 0, \quad \|v_k\|_{L^p(\Omega)} = 1, \quad \|Dv_k\|_{L^p(\Omega)} < \frac{1}{k}, \qquad \text{for } k \in \{1, 2, \ldots\}. \tag{1.10}$$

Since the sequence $(v_k)_{k \geqslant 1}$ is bounded in $W^{1,p}(\Omega)$, if $p < \infty$, we can use the Rellich–Kondrachov's compactness theorem and find a subsequence that converges in $L^p(\Omega)$ to some function $v$. If $p > n$, then the functions $v_k$ are uniformly bounded and Hölder continuous. Using Ascoli–Arzela's compactness theorem, we can thus find a subsequence that converges in $L^\infty(\Omega)$ to some function $v$.

By (1.10), the sequence of weak gradients also converges, namely $\nabla v_k \to 0$ in $L^p(\Omega)$. By Lemma 4.2, the zero function is the weak gradient of the limit function $v$.

We now have

$$\fint_\Omega v\,\mathrm{d}x = \lim_{k\to\infty} \fint_\Omega v_k\,\mathrm{d}x = 0.$$

Moreover, since $\nabla v = 0 \in L^p(\Omega)$, then the function $v$ must be constant on the connected set $\Omega$; hence $v(x) = 0$ for a.e. $x \in \Omega$. But this is in contradiction with

$$\|v\|_{L^p(\Omega)} = \lim_{k\to\infty} \|v_k\|_{L^p(\Omega)} = 1.$$

□

**1.8. Dual Sobolev spaces.** We can now define Sobolev spaces of negative order by duality using $W_0^{k,p}(\Omega)$.

Definition 4.8. *Let $k > 0$ and $p \in (1,\infty)$. Let $\Omega$ be an open set in $\mathbb{R}^n$. We define the space $W^{-k,p}(\Omega) := \left(W_0^{k,p'}(\Omega)\right)'$ with $\frac{1}{p} + \frac{1}{p'} = 1$ (for $p = 2$, we write $H^{-k}(\Omega) := W^{-k,2}(\Omega)$), equipped with the norm*

$$\|T\|_{W^{-k,p}(\Omega)} := \sup_{w\in W_0^{k,p'}(\Omega)} \frac{|\langle T, w\rangle|}{\|w\|_{W^{k,p'}(\Omega)}}.$$

Identifying $L^p(\Omega)$ with the dual space of $L^{p'}(\Omega)$, we infer that $L^p(\Omega) \hookrightarrow W^{-k,p}(\Omega)$. Moreover, one can check that any element $T \in W^{-k,p}(\Omega)$ is a distribution.

Example 4.8. *The Dirac mass at a point $x \in \Omega$ belongs to $W^{-k,p}(\Omega)$ if $kp' > n$.*

Theorem 4.18. *Let $p \in (1,\infty)$. Let $\Omega$ be an open bounded set in $\mathbb{R}^n$. For all $f \in W^{-1,p}(\Omega)$, there exist functions $\{g_i\}_{i=1,\dots,n}$, belonging to $L^{p'}(\Omega)$, such that $\|f\|_{W^{-1,p}(\Omega)} = \max_{i\in\{0,\dots,n\}} \|g_i\|_{L^{p'}(\Omega)}$ and*

$$\langle f, v\rangle = \int_\Omega g_0 v\,\mathrm{d}x + \sum_{i=1}^n \int_\Omega g_i \partial_{x_i} v\,\mathrm{d}x, \qquad \textit{for all } v \in W_0^{1,p}(\Omega).$$

*More generally, for all $m \in \mathbb{N}$, we have $f \in W^{-m,p}(\Omega)$ if and only if $f = \sum_{|\alpha|\leqslant m} \partial^\alpha g_\alpha$ for some $g_\alpha \in L^{p'}(\Omega)$.*

For future use, we focus on the particular case $k = 1$ and $p = 2$. The space of bounded linear maps $f : H_0^1(\Omega) \to \mathbb{R}$ is denoted by $H^{-1}(\Omega) = (H_0^1(\Omega))'$, and the action of $f \in H^{-1}(\Omega)$ on $\phi \in H_0^1(\Omega)$ by $\langle f, \phi\rangle$. The norm of $f \in H^{-1}(\Omega)$ is given by

$$\|f\|_{H^{-1}} = \sup\left\{\frac{|\langle f, \phi\rangle|}{\|\phi\|_{H_0^1}} : \phi \in H_0^1, \phi \neq 0\right\}.$$

A function $f \in L^2(\Omega)$ defines a linear functional $F_f \in H^{-1}(\Omega)$ by

$$\langle F_f, v\rangle = \int_\Omega f v\,\mathrm{d}x = (f, v)_{L^2} \quad \text{for all } v \in H_0^1(\Omega)$$

Here, $(\cdot,\cdot)_{L^2}$ denotes the standard inner product on $L^2(\Omega)$. The functional $F_f$ is bounded on $H_0^1(\Omega)$ with $\|F_f\|_{H-1} \leqslant \|f\|_{L^2}$ since, by Cauchy-Schwarz' inequality,[11]

$$|\langle F_f, v\rangle| \leqslant \|f\|_{L^2}\|v\|_{L^2} \leqslant \|f\|_{L^2}\|v\|_{H_0^1}$$

We identify $F_f$ with $f$, and write both simply as $f$. Such linear functionals are, however, not the only elements of $H^{-1}(\Omega)$. As shown in Theorem 4.18, $H^{-1}(\Omega)$ may be identified with the space of distributions on $\Omega$ that are sums of first-order distributional derivatives of functions in $L^2(\Omega)$. Thus, after identifying functions with regular distributions, we have the following triple of Hilbert spaces

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega), \quad H^{-1}(\Omega) = (H_0^1(\Omega))'$$

[11] Also called Cauchy–Bunyakovsky–Schwarz' inequality, after Augustin-Louis Cauchy [Cau21], Viktor Bunyakovsky [Bun59], and Hermann Schwarz [Sch88].

Moreover, if $f \in L^2(\Omega) \subset H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$, then

$$\langle f, u \rangle = (f, u)_{L^2},$$

so the duality pairing coincides with the $L^2$-inner product when both are defined.

## 2. Weak solutions of elliptic PDEs

Let us consider the Dirichlet problem for the Laplacian with homogeneous boundary conditions on a bounded domain $\Omega$ in $\mathbb{R}^n$,

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

First, suppose that the boundary of $\Omega$ is smooth and $u, f : \bar{\Omega} \to \mathbb{R}$ are smooth functions. Multiplying the PDE $-\Delta u = f$ by a test function $\phi \in \mathcal{D}(\Omega)$, integrating the result over $\Omega$, and using the divergence theorem, we get

$$\int_\Omega Du \cdot D\phi \, \mathrm{d}x = \int_\Omega f\phi \, \mathrm{d}x \qquad \text{for all } \phi \in C_c^\infty(\Omega). \tag{2.2}$$

The boundary terms vanish because $\phi = 0$ on the boundary. Conversely, if $f$ and $\Omega$ are smooth, then any smooth function $u$ that satisfies (2.2) is a solution of the Dirichlet problem (2.1).

Next, we formulate weaker assumptions under which (2.2) makes sense. By the Cauchy–Schwartz inequality, the integral on the left-hand side of (2.2) is finite if $Du$ belongs to $L^2(\Omega)$, so we suppose that $u \in H^1(\Omega)$. Moreover, we impose the boundary condition $u = 0$ on $\partial\Omega$ of (2.1) in a weak sense by requiring that $u \in H_0^1(\Omega)$. The left hand side of (2.2) then extends by continuity to $\phi \in H_0^1(\Omega) = \overline{C_c^\infty(\Omega)}$. The right hand side of (2.2) is well-defined for all $\phi \in H_0^1(\Omega)$ if $f \in L^2(\Omega)$, but this is not the most general $f$ for which it makes sense; we can define the right-hand for any $f$ in the dual space of $H_0^1(\Omega)$.

Definition 4.9 (Weak solutions of the Dirichlet problem (2.1)). *Let $\Omega$ be an open set in $\mathbb{R}^n$ and $f \in H^{-1}(\Omega)$. A function $u : \Omega \to \mathbb{R}$ is a weak solution of (2.1) if $u \in H_0^1(\Omega)$ and*

$$\int_\Omega Du \cdot D\phi \, \mathrm{d}x = \langle f, \phi \rangle \qquad \text{for all } \phi \in H_0^1(\Omega).$$

Remark 4.10. *If $\Omega$ is smooth and $g : \partial\Omega \to \mathbb{R}$ is a function on the boundary that is in the range of the trace map $\tau : H^1(\Omega) \to L^2(\partial\Omega)$, say $g = \tau w$, then we obtain a weak formulation of the inhomogeneous Dirichet problem*

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ u = g, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

*by requiring $u - w \in H_0^1(\Omega)$ instead of $u \in H_0^1(\Omega)$. The definition is otherwise the same. One can prove that the range of the trace map on $H^1(\Omega)$ for a smooth domain $\Omega$ is the fractional-order Sobolev space $H^{1/2}(\partial\Omega)$.*

Remark 4.11 (Distributional solutions of the Poisson equation). *Let us comment on some other ways to define weak solutions of Poisson's equation. If we integrate by parts again in (2.2), we find that every smooth solution $u$ of (2.1) satisfies*

$$-\int_\Omega u\Delta\phi \, \mathrm{d}x = \int_\Omega f\phi \, \mathrm{d}x \qquad \text{for all } \phi \in C_c^\infty(\Omega). \tag{2.4}$$

*This condition makes sense without any differentiability assumptions on $u$, and we can define a locally integrable function $u \in L^1_{loc}(\Omega)$ to be a weak solution of $-\Delta u = f$ for $f \in L^1_{\mathrm{loc}}(\Omega)$ if it satisfies (2.4). One problem with using this definition is that general functions $u \in L^p(\Omega)$ do not have enough regularity to make sense of their boundary values on $\partial\Omega$.*

*More generally, we can define distributional solutions $T \in \mathcal{D}'(\Omega)$ of Poisson's equation $-\Delta T = f$ with $f \in \mathcal{D}'(\Omega)$ by*

$$-\langle T, \Delta\phi \rangle = \langle f, \phi \rangle \quad \text{for all } \phi \in C_c^\infty(\Omega).$$

*While these definitions appear more general, owing to some elliptic regularity results they turn out not to extend the class of weak solutions we consider in Definition 4.9 if $f \in H^{-1}(\Omega)$.*

## 3. Existence of weak solutions for elliptic PDEs via Riesz' representation theorem

In this section, we establish the existence and uniqueness of weak solutions to (2.1).

THEOREM 4.19 (Well-posedness of weak solutions to the Dirichlet problem (2.1)). *Suppose that $\Omega$ is an open set in $\mathbb{R}^n$ that is bounded in some direction and $f \in H^{-1}(\Omega)$. Then there is a unique weak solution $u \in H_0^1(\Omega)$ of $-\Delta u = f$ in the sense of Definition 4.9.*

The main tool is Riesz' representation theorem, which states that a Hilbert space can be identified with its dual.[12]

THEOREM 4.20 (Riesz representation of linear functionals). *Let $(H, (\cdot,\cdot))$ be a Hilbert space.*

*(1) For every $x \in H$, the map $y \mapsto (y, x)$ is a continuous linear functional on $H$.*
*(2) Let $y \mapsto Ay$ be a continuous linear functional on $H$. Then there exists a unique element $a \in H$ such that $Ay = (y, a)$ for every $y \in H$.*

PROOF OF THEOREM 4.19. We equip $H_0^1(\Omega)$ with the inner product (1.9). Then, since $\Omega$ is bounded in some direction, the resulting norm is equivalent to the standard norm, and $f$ is a bounded linear functional on $\big(H_0^1(\Omega), (\cdot,\cdot)_0\big)$. By Riesz' representation theorem, there exists a unique $u \in H_0^1(\Omega)$ such that

$$(u, \phi)_0 = \langle f, \phi\rangle \quad \text{for all } \phi \in H_0^1(\Omega),$$

which is equivalent to the condition that $u$ is a weak solution. □

The same approach works for other *symmetric* linear elliptic PDEs, as we will see below.

**3.1. Inhomogeneous Dirichlet problem.** Let $f \in H^{-1}(\Omega)$ and $g \in H^1(\Omega)$. For the inhomogeneous Dirichlet problem (2.3), we consider the closed convex set

$$K(\Omega) = H_0^1(\Omega) + g := \{w \in H^1(\Omega) : w - g \in H_0^1(\Omega)\}.$$

A function $u \in K(\Omega)$ is a weak solution of (2.3) if

$$\int_\Omega \nabla u \cdot \nabla\phi \,\mathrm{d}x = \int_\Omega f\phi \,\mathrm{d}x, \qquad \text{for all } \phi \in H_0^1(\Omega).$$

REMARK 4.12. *Let $f \in C(\Omega)$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$. Then $u$ is a weak solution of (2.3) if and only if $u$ is a classical solution of (2.3).*

To show the existence and uniqueness of weak solutions to (2.3), we apply Riesz' representation theorem to the linear functional $G : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$G(\phi) := \int_\Omega f\phi - \int_\Omega \nabla g \cdot \nabla\phi.$$

Then there exists a unique $w \in H_0^1(\Omega)$ such that $G(\phi) = (\nabla w, \cdot\nabla\phi)_0$. Then $u := w + g$ is the unique function in $K(\Omega)$ such that

$$\int_\Omega \nabla v \cdot \nabla\phi = \int_\Omega f\phi,$$

i.e., the unique weak solution of (2.3).

REMARK 4.13. *Uniqueness can be established a priori via energy methods. Let $u_1, u_2 \in K(\Omega)$. Then, for all $\phi \in H_0^1(\Omega)$, we have*

$$\int_\Omega \nabla(u_1 - u_2) \cdot \nabla\phi = 0.$$

*In particular, letting $\phi = u_1 - u_2$, we conclude $u_1 = u_2$ a.e.*

[12] This result is also known as *Riesz–Fréchet representation theorem*, after Frigyes Riesz [Rie07] and Maurice René Fréchet [Fré07].

**3.2. Homogeneous Dirichlet problem for $-\Delta + \mathrm{I}$.** Consider the Dirichlet problem

$$\begin{cases} -\Delta u + u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Then $u \in H_0^1(\Omega)$ is a weak solution if

$$\int_\Omega (Du \cdot D\phi + u\phi)\,\mathrm{d}x = \langle f, \phi \rangle \qquad \text{for all } \phi \in H_0^1(\Omega).$$

This is equivalent to the condition that

$$(u, \phi)_1 = \langle f, \phi \rangle \qquad \text{for all } \phi \in H_0^1(\Omega),$$

where $(\cdot,\cdot)_1$ is the standard inner product on $H_0^1(\Omega)$. Thus, Riesz' representation theorem implies the existence of a unique weak solution.

Remark 4.14. *Note that in this example we do not use the Poincaré inequality, so the result applies to arbitrary open sets, including $\Omega = \mathbb{R}^n$. In that case, $H_0^1(\mathbb{R}^n) = H^1(\mathbb{R}^n)$, and we get a unique solution $u \in H^1(\mathbb{R}^n)$ of $-\Delta u + u = f$ for every $f \in H^{-1}(\mathbb{R}^n)$. Moreover, using the standard norms, we have $\|u\|_{H^1} = \|f\|_{H^{-1}}$. Thus the operator $-\Delta + I$ is an isometry of $H^1(\mathbb{R}^n)$ onto $H^{-1}(\mathbb{R}^n)$.*

**3.3. Homogeneous Dirichlet problem for $-\Delta + \mu\mathrm{I}$.** Let $\mu > 0$. A function $u \in H_0^1(\Omega)$ is a weak solution of

$$\begin{cases} -\Delta u + \mu u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

if

$$(u, \phi)_\mu = \langle f, \phi \rangle \qquad \text{for all } \phi \in H_0^1(\Omega),$$

where

$$(u, v)_\mu = \int_\Omega (\mu uv + Du \cdot Dv)\,\mathrm{d}x$$

The norm $\|\cdot\|_\mu$ associated with this inner product is equivalent to the standard one, since

$$\frac{1}{C}\|u\|_\mu^2 \leqslant \|u\|_1^2 \leqslant C\|u\|_\mu^2$$

where $C := \max\{\mu, 1/\mu\}$. We therefore again get the existence of a unique weak solution from Riesz' representation theorem.

Example 4.9. *Consider the last example for $\mu < 0$. If we have a Poincaré inequality $\|u\|_{L^2} \leqslant C\|Du\|_{L^2}$ for $\Omega$, which is the case if $\Omega$ is bounded in some direction, then*

$$(u, u)_\mu = \int_\Omega \left(\mu u^2 + |Du|^2\right)\,\mathrm{d}x \geqslant (1 - C|\mu|) \int_\Omega |Du|^2\,\mathrm{d}x$$

*Thus $\|u\|_\mu$ defines a norm on $H_0^1(\Omega)$ that is equivalent to the standard norm if $-1/C < \mu < 0$, and we get a unique weak solution in this case also, provided that $|\mu|$ is sufficiently small.*

*For bounded domains, the Dirichlet Laplacian has an infinite sequence of real eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ such that there exists a nonzero solution $u \in H_0^1(\Omega)$ of $-\Delta u = \lambda_n u$. The best constant in Poincaré's inequality can be shown to be the minimum eigenvalue $\lambda_1$, and this method does not work if $\mu \leqslant -\lambda_1$. For $\mu = -\lambda_n$, a weak solution of does not exist for every $f \in H^{-1}(\Omega)$, and if one does exist it is not unique since we can add to it an arbitrary eigenfunction. Thus, not only does the method fail, but the conclusion of Theorem 4.19 may be false.*

**3.4. Homogeneous Dirichlet problem for symmetric elliptic operators.** Let us consider the operator

$$Lu = -\sum_{i,j=1}^n \partial_{x_i}\left(a_{ij}\partial_{x_j} u\right),$$

where the coefficients are assumed to be bounded, symmetric ($a_{ij} = a_{ji}$), and satisfy the uniform ellipticity condition. That is, for some $\theta > 0$,

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geqslant \theta|\xi|^2 \quad \text{for all } x \in \Omega, \text{ and all } \xi \in \mathbb{R}^n.$$

The function $u \in H_0^1(\Omega)$ will be a weak solution of the Dirichlet problem for this operator,

$$\begin{cases} Lu = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

if

$$a(u, \phi) = \langle f, \phi \rangle \quad \text{for all } \phi \in H_0^1(\Omega),$$

where $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is the symmetric bilinear form associated with the operator, and is given by

$$a(u, v) := \sum_{i,j=1}^{n} \int_\Omega a_{ij} \partial_{x_j} u \partial_{x_i} v \, \mathrm{d}x$$

If $\Omega$ is bounded in some direction, then boundedness of $a_{ij}$, uniform ellipticiy, and Poincaré's inequality will imply that the symmetric bilinear form $a$ defines an inner product on $H_0^1(\Omega)$, with the induced norm being equivalent to the standard norm of $H_0^1(\Omega)$. This will again imply that $f \in H^{-1}$ is a bounded linear functional on the Hilbert space $\big(H_0^1(\Omega), a\big)$, and hence Riesz' representation theorem will imply the existence of a unique weak solution of the Dirichlet problem for this operator.

REMARK 4.15. *The bilinear form $a$ of course arises from integration by parts of the left hand side of the equation after multiplying by the function $v$. Thus, having the derivative in front of the entire term $a_{ij}\partial_{x_j} u$ is crucial, since we are not assuming that the coefficients $a_{ij}$ are weakly differentiable. In such cases, we will say that the elliptic operator is in the* divergence form.

**3.5. Homogeneous Neumann problem for $-\Delta$+I.** Let $\Omega$ be an open, connected, bounded set with $C^1$ boundary. If $f \in L^2(\Omega)$, $u$ is a weak solution of

$$\begin{cases} -\Delta u + u = f, & x \in \Omega, \\ \partial_\nu u = 0, & x \in \partial\Omega, \end{cases} \tag{3.1}$$

if

$$\int_\Omega \nabla u \cdot \nabla \phi + \int_\Omega u\phi = \int_\Omega f\phi, \qquad \text{for all } \phi \in H^1(\Omega).$$

This is equivalent to the condition that

$$(u, \phi)_1 = \langle f, \phi \rangle \qquad \text{for all } \phi \in H^1(\Omega),$$

where $(\cdot, \cdot)_1$ is the standard inner product on $H^1(\Omega)$. Riesz' representation theorem yields the existence of a unique weak solution.

REMARK 4.16. *Let $f \in C(\Omega)$. If $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a classical solution, then it is a weak solution of (3.1).*

*Viceversa, if $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a weak solution, then it is a classical solution of (3.1). To check this, let us consider a test function $v \in \mathcal{D}(\Omega) \subset H^1(\Omega)$. Since $u$ is a weak solution, using Gauss–Green's formulas, we have*

$$\int_\Omega fv = \int_\Omega \nabla v \cdot \nabla u + \int_\Omega uv = \int_\Omega (-\Delta u)v + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, \mathrm{d}S + \int_\Omega uv = \int_\Omega (-\Delta u)v + \int_\Omega uv.$$

*In the last equality, we used the assumption on the support of $v$. Therefore, for every $v \in \mathcal{D}(\Omega)$, we have*

$$\int_\Omega (-\Delta u + u - f)v = 0.$$

*By the fundamental lemma of the calculus of variations, $(-\Delta u + u - f) = 0$ almost everywhere in $\Omega$, but since $u$ and $f$ are continuous, this equality holds for every $x \in \Omega$.*

*It remains to prove that*

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

*Let $v \in \mathcal{C}^\infty(\bar{\Omega}) \subset H^1(\Omega)$. We know that $u$ satisfies $(-\Delta u + u - f) = 0$ and is also a weak solution of the Neumann problem (3.1). From this, we deduce that*

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega (-\Delta u)v + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, \mathrm{d}S = \int_\Omega (-u + f)v + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, \mathrm{d}S.$$

*That is,*

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \,\mathrm{d}S = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega uv - \int_\Omega fv = 0.$$

*It can be shown that the set* $\{v_{|\partial\Omega} : v \in \mathcal{C}^\infty(\bar{\Omega})\}$ *is dense in* $L^2(\partial\Omega)$. *Therefore, by continuity,*

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \,\mathrm{d}S = 0 \quad \textit{for all } v \in L^2(\partial\Omega),$$

*since* $\frac{\partial u}{\partial \nu} \in L^2(\partial\Omega)$ *is orthogonal to a dense subspace. Ultimately,* $\partial_\nu u = 0$ *on* $\partial\Omega$.

*It is said that the Neumann boundary value problem has* natural boundary conditions, *because the space in which the weak solution is defined does not explicitly contain information about* $\partial_\nu u$, *but this condition is automatically satisfied.*

**3.6. Homogeneous Neumann problem for the Poisson equation.** Let $\Omega$ be an open, connected, bounded set with $C^1$ boundary. Let $f \in L^2(\Omega)$ and consider

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ \partial_\nu u = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

Let $f \in L^2(\Omega)$. We say that $u \in H^1(\Omega)$ is a weak solution of (3.2) if

$$\int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega f\phi, \qquad \text{for all } \phi \in H^1(\Omega).$$

THEOREM 4.21. *If a weak solution of (3.2) exists, then it is unique up to additive constants.*

PROOF. Let $u_1$, $u_2 \in H^1(\Omega)$ be two weak solutions of (3.2). Then, for every $\phi \in H^1(\Omega)$, we have

$$\int_\Omega \nabla(u_1 - u_2) \cdot \nabla\phi = 0.$$

Choosing, in particular, $\phi = u_1 - u_2$, yields $\nabla(u_1 - u_2) = 0$ a.e., and so $u_1 = u_2 + c$ (for some constant $c \in \mathbb{R}$), since $\Omega$ is connected. ☐

THEOREM 4.22. *There exists a weak solution of (3.2) if and only if* $⨍_\Omega f(x)\,\mathrm{d}x = 0$

PROOF. Step 1. If a weak solution exists, then we pick $\phi \equiv$ const. in the weak formulation and deduce $\int_\Omega f(x)\,\mathrm{d}x = 0$.

Step 2. Let us suppose $⨍_\Omega f(x)\,\mathrm{d}x = 0$ and define

$$V = \left\{ u \in H^1(\Omega) : ⨍_\Omega u \,\mathrm{d}x = 0 \right\}.$$

This is a closed subspace of $H^1(\Omega)$ (for example, one can see this as $V$ is the kernel of the linear continuous functional $\langle 1, \cdot\rangle \in L^2(\Omega)$. By Poincaré–Wirtinger's inequality, $V$ is a Hilbert space with norm $\|\nabla u\|_{L^2(\Omega)}$. Since weak solutions are unique up to an additive constant, we may assume that they lie all in $V$ (otherwise, we subtract the average). It suffices to test weak solutions with $\phi \in V$. Indeed, since $⨍_\Omega f = 0$, we have

$$\int \phi f = \int \left( \phi - ⨍ \phi \right) f.$$

Applying Riesz' representation theorem on the Hilbert space $V$, we get existence of a unique weak solution of (3.2) in $V$. ☐

## 4. Existence of weak solutions for elliptic PDEs via variational methods

We have established the existence of a weak solution by use of Riesz' representation theorem. In this section, we use a different approach, via *variational methods*. The Riesz representation theorem is, however, typically proved by a similar argument to the one used in the *direct method of the calculus of variations*, so in essence the proofs are equivalent.

DEFINITION 4.10. *A functional $J : X \to \mathbb{R}$ on a Banach space $X$ is* differentiable *at $x \in X$ if there is a bounded linear functional $A : X \to \mathbb{R}$ such that*

$$\lim_{h\to 0} \frac{|J(x+h) - J(x) - Ah|}{\|h\|_X} = 0$$

*If $A$ exists, then it is unique, and it is called the* differential *of $J$ at $x$, denoted $DJ(x) = A$.*

This definition expresses the basic idea of a differentiable function as one which can be approximated locally by a linear map. If $J$ is differentiable at every point of $X$, then $DJ : X \to X^*$ maps $x \in X$ to the linear functional $DJ(x) \in X^*$ that approximates $J$ near $x$.

A weaker notion of differentiability is the existence of *directional derivatives*

$$\delta J(x;h) = \lim_{\epsilon\to 0} \left[\frac{J(x+\epsilon h) - J(x)}{\epsilon}\right] = \frac{\mathrm{d}}{\mathrm{d}\epsilon} J(x+\epsilon h)\bigg|_{\epsilon=0}.$$

If the directional derivative at $x$ exists for every $h \in X$ and is a bounded linear functional on $h$, then $\delta J(x;h) = \delta J(x)h$ where $\delta J(x) \in X^*$. We call $\delta J(x)$ the *Gâteaux derivative* of $J$ at $x$. The derivative $DJ$ is then called the *Fréchet derivative* to distinguish it from the directional or Gâteaux derivative. If $J$ is differentiable at $x$, then it is Gâteaux-differentiable at $x$ and $DJ(x) = \delta J(x)$, but the converse is not true.

EXAMPLE 4.10. *Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0,0) = 0$ and*

$$f(x,y) = \left(\frac{xy^2}{x^2+y^4}\right)^2 \qquad \text{if } (x,y) \neq (0,0).$$

*Then $f$ is Gâteaux-differentiable at $0$, with $\delta f(0) = 0$, but $f$ is not Fréchet differentiable at $0$.*

If $J : X \to \mathbb{R}$ attains a local minimum at $x \in X$ and $J$ is differentiable at $x$, then for every $h \in X$ the function $J_{x;h} : \mathbb{R} \to \mathbb{R}$ defined by $J_{x;h}(t) = J(x+th)$ is differentiable at $t = 0$ and attains a minimum at $t = 0$. It follows that

$$\frac{\mathrm{d}J_{x;h}}{\mathrm{d}t}(0) = \delta J(x;h) = 0 \qquad \text{for every } h \in X.$$

Hence $DJ(x) = 0$.

We say that an elliptic problem is in *variational form* if its weak solutions are critical points of a suitable functional.

If we have a variational problem, in the *direct method of the calculus of variations*, we prove the existence of a minimizer of $J$ by showing that a minimizing sequence $\{u_n\}$ converges in a suitable sense to a minimizer $u$. The direct method of the calculus of variations is encoded in the proof of the (generlized) Weierstrass theorem.[13]

THEOREM 4.23 ((Generlized) Weierstrass' theorem). *Let $X$ be a reflexive space, $F : X \to \mathbb{R}$ be a coercive and weakly lower-semicontinuous functional. Then $\inf_X F > -\infty$ and it is a minimum., i.e., there exists $\bar{x} \in X$ such that $F(\bar{x}) = \inf_X F$. Moreover, if $F$ is strictly convex, then the minimum point is unique.*

REMARK 4.17. *Let $X$ be a Banach space, we recall that $F : X \to ]-\infty, +\infty]$ is called*

- *weakly lower-semicountinous if, for every $x \in X$ and every $x_n \to x$, we have $\liminf_{n\to\infty} F(x_n) \geqslant F(x)$;*
- *weakly lower-semicountinous if, for every $x \in X$ and every $x_n \rightharpoonup x$, we have $\liminf_{n\to\infty} F(x_n) \geqslant F(x)$;*
- *coercive if $\lim_{\|x\|_X\to\infty} F(x) = +\infty$;*
- *convex if, for every $x$, $y \in X$ and every $t \in [0,1]$, we have $F(tx + (1-t)x) \leqslant tF(x) + (1-t)F(x)$.*

*We also recall that, if $F$ is convex and lower-semicountinuous, then it is weakly lower-semicontinuous.*

[13] Named after Karl Theodor Wilhelm Weierstrass.

**4.1. Homogeneous Dirichlet problem for the Poisson equation.** Given $f \in H^{-1}(\Omega)$, we define a quadratic functional $J : H_0^1(\Omega) \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_\Omega |Du|^2 \, dx - \langle f, u\rangle. \tag{4.1}$$

PROPOSITION 4.2. *The functional $J : H_0^1(\Omega) \to \mathbb{R}$ in (4.1) is well-defined and differentiable. Its derivative $DJ(u) : H_0^1(\Omega) \to \mathbb{R}$ at $u \in H_0^1(\Omega)$ is given by*

$$DJ(u)h = \int_\Omega Du \cdot Dh \, dx - \langle f, h\rangle \quad \text{for } h \in H_0^1(\Omega).$$

PROOF. Given $u \in H_0^1(\Omega)$, define the linear map $A : H_0^1(\Omega) \to \mathbb{R}$ by

$$Ah = \int_\Omega Du \cdot Dh \, dx - \langle f, h\rangle.$$

Then $A$ is bounded, with $\|A\| \leqslant \|Du\|_{L^2} + \|f\|_{H^{-1}}$, since

$$|Ah| \leqslant \|Du\|_{L^2}\|Dh\|_{L^2} + \|f\|_{H^{-1}}\|h\|_{H_0^1} \leqslant (\|Du\|_{L^2} + \|f\|_{H^{-1}})\,\|h\|_{H_0^1}$$

For $h \in H_0^1(\Omega)$, we have

$$J(u+h) - J(u) - Ah = \frac{1}{2}\int_\Omega |Dh|^2 \, dx.$$

It follows that

$$|J(u+h) - J(u) - Ah| \leqslant \frac{1}{2}\|h\|_{H_0^1}^2,$$

and therefore

$$\lim_{h\to 0} \frac{|J(u+h) - J(u) - Ah|}{\|h\|_{H_0^1}} = 0$$

which proves that $J$ is differentiable on $H_0^1(\Omega)$ with $DJ(u) = A$. □

Note that $DJ(u) = 0$ if and only if $u$ is a weak solution of the Dirichlet problem Poisson's equation

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{4.2}$$

in the sense of Definition 4.9. Thus, we have the following result.

COROLLARY 4.3. *If $J : H_0^1(\Omega) \to \mathbb{R}$ defined in (4.1) attains a minimum at $u \in H_0^1(\Omega)$, then $u$ is a weak solution of $-\Delta u = f$.*

PROOF. Let us define $g(t) = J(u + tv)$ for $v \in H_0^1(\Omega)$. We know that $g$ has a minimum point at $t = 0$. In particular,

$$\begin{aligned} g(t) = J(u+tv) &= \frac{1}{2}\int_\Omega |\nabla(u+tv)|^2 \, dx - \int_\Omega f(u+tv)\, dx \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx + \frac{t^2}{2}\int_\Omega |\nabla v|^2 \, dx + t\int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega fu \, dx - t\int_\Omega fv \, dx. \end{aligned}$$

Then, the condition $g'(0) = 0$ ensures

$$\int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega fv \, dx = 0.$$

□

Finally, we can prove that $J$ has one and only one minimum. Indeed, we can show that $J$ is coercive, strictly convex, and lower-semicontinuous (which proves the claim owing to Theorem 4.23).

**Coercivity:** We aim to estimate $J(v)$ from below:

$$J(v) = \frac{1}{2}\|v\|_{H_0^1(\Omega)}^2 - \int_\Omega fv \geqslant \frac{1}{2}\|v\|_{H_0^1(\Omega)}^2 - \|f\|_2\|v\|_2 \geqslant \frac{1}{2}\|v\|_{H_0^1(\Omega)}^2 - C_P\|f\|_2\|v\|_{H_0^1(\Omega)},$$

where $C_P$ is the constant from Poincaré's inequality. Thus, $J(v) \to +\infty$ as $\|v\|_{H_0^1(\Omega)} \to +\infty$ (since $J(v)$ contains a quadratic polynomial in $\|v\|_{H_0^1(\Omega)}$).

**Strict Convexity:** Using the strict convexity of the single-variable function $f(x) = x^2$ (which follows from $f'' = 2 > 0$), we have:

$$\begin{aligned}
&J\left(tv_1 + (1-t)v_2\right) - \left(tJ(v_1) + (1-t)J(v_2)\right) \\
&= \frac{1}{2}\int_\Omega \left|t\nabla v_1 + (1-t)\nabla v_2\right|^2 \,\mathrm{d}x - \int_\Omega f\left(tv_1 + (1-t)v_2\right) \,\mathrm{d}x \\
&\quad - \frac{t}{2}\int_\Omega |\nabla v_1|^2 \,\mathrm{d}x - \frac{1-t}{2}\int_\Omega |\nabla v_2|^2 \,\mathrm{d}x + t\int_\Omega f v_1 \,\mathrm{d}x + (1-t)\int_\Omega f v_2 \,\mathrm{d}x \\
&< \frac{1}{2}\int_\Omega t|\nabla v_1|^2 + (1-t)|\nabla v_2|^2 \,\mathrm{d}x - \frac{t}{2}\int_\Omega |\nabla v_1|^2 \,\mathrm{d}x - \frac{1-t}{2}\int_\Omega |\nabla v_2|^2 \,\mathrm{d}x = 0.
\end{aligned}$$

**Lower Semicontinuity:** Since the functional $J$ is continuous, it is lower semicontinuous with respect to the strong topology. Furthermore, since it is convex, it is also weakly lower semicontinuous.

Thus, by the Weierstrass theorem, there exists a unique $u \in H_0^1(\Omega)$ that minimizes $J$.

**4.2. Inhomogeneous Dirichlet problem for the Poisson equation.** We now turn to the inhomogeneous Dirichlet problem for the Poisson equation:

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ u = g, & x \in \partial\Omega, \end{cases} \tag{4.3}$$

We will use the notation of Section 3.1.

THEOREM 4.24. *Let $f \in L^2(\Omega)$ and $g \in H^1(\Omega) \cap C(\bar{\Omega})$. Then there exists one and only one weak solution of (4.3).*

PROOF. Let $J : K \to \mathbb{R}$ be defined as

$$J(u) = \frac{1}{2}\int |\nabla u|^2 \,\mathrm{d}x - \int f u \,\mathrm{d}x.$$

To this functional, we associate the functional $\tilde{J} : H_0^1(\Omega) \to \mathbb{R}$, defined by

$$\tilde{J}(w) := J(w + g).$$

We aim to prove that $\tilde{J}$ has a unique minimum point. Explicitly, we have

$$\tilde{J}(w) = J(w+\varphi) = \frac{1}{2}\int_\Omega |\nabla w|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla \varphi|^2 \,\mathrm{d}x + \int_\Omega \nabla w \cdot \nabla \varphi \,\mathrm{d}x - \int_\Omega f w \,\mathrm{d}x - \int_\Omega f\varphi \,\mathrm{d}x.$$

Since $\tilde{J}$ is defined on $H_0^1(\Omega)$, which is a reflexive space, we apply Weierstrass' theorem to find the minimum points by proving that $\tilde{J}$ is coercive and weakly lower semi-continuous. In fact, we will find a unique minimum of $\tilde{J}$, which, when shifted back, will correspond to the unique minimum of $J$.

To prove the coercivity of $\tilde{J}$, we use Hölder's inequality:

$$-\int_\Omega \nabla w \cdot \nabla\varphi \,\mathrm{d}x \leqslant \left|\int_\Omega \nabla w \cdot \nabla\varphi \,\mathrm{d}x\right| \leqslant \|\nabla w\|_2 \|\nabla \varphi\|_2.$$

From Poincaré's inequality, it follows that

$$\int_\Omega f w \,\mathrm{d}x \leqslant \|f\|_2\|w\|_2 \leqslant C\|f\|_2\|\nabla w\|_2.$$

Changing the signs in the previous inequalities, we get

$$|\tilde{J}(w)| \geqslant \frac{1}{2}\|\nabla w\|_2^2 + \frac{1}{2}\|\nabla\varphi\|_2^2 - \|\nabla w\|_2\|\nabla\varphi\|_2 - C\|f\|_2\|\nabla w\|_2 - \int_\Omega f\varphi \,\mathrm{d}x.$$

Thus, as $\|\nabla w\|_2 \to +\infty$, we have $|\tilde{J}(w)| \to +\infty$, since $\tilde{J}(w)$ contains a quadratic polynomial in $\|\nabla w\|_2$.

Next, we show that $\tilde{J}$ is weakly lower semicontinuous. Indeed:

- The term $\frac{1}{2}\int_\Omega |\nabla w|^2 \,\mathrm{d}x$ is weakly lower semicontinuous because it represents the norm squared of the space in question (proof of this fact omitted).
- The terms $\int_\Omega \nabla w \cdot \nabla\varphi \,\mathrm{d}x - \int_\Omega f w \,\mathrm{d}x$ are weakly continuous because they are linear and continuous.

- The terms $\frac{1}{2}\int_\Omega |\nabla\varphi|^2 \, \mathrm{d}x - \int_\Omega f\varphi \, \mathrm{d}x$ are constant.

Hence, by Weierstrass' theorem, $\tilde{J}$ has a minimum point. We now prove the strict convexity of $\tilde{J}$ to ensure the uniqueness of this minimum. Since $\tilde{J}$ contains constant terms and linear terms, its strict convexity reduces to the strict convexity of the squared $H_0^1$-norm. Specifically:

$$\begin{aligned}
&\tilde{J}\left(tw_1 + (1-t)w_2\right) - \left(t\tilde{J}(w_1) + (1-t)\tilde{J}(w_2)\right) \\
&= \frac{1}{2}\left[\|tw_1 + (1-t)w_2\|^2_{H_0^1(\Omega)} - t\|w_1\|^2_{H_0^1(\Omega)} - (1-t)\|w_2\|^2_{H_0^1(\Omega)}\right].
\end{aligned}$$

Let $w_0$ be the minimum obtained from Weierstrass' theorem for $\tilde{J}$ on $H_0^1(\Omega)$. Then $u_0 = w_0 + \varphi$ is the minimum of the functional $J$ on $K$. Clearly, $u_0 \in K$. We find that $u_0$ is a weak solution, meaning that for every $v \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u_0 \cdot \nabla v \, \mathrm{d}x = \int_\Omega fv \, \mathrm{d}x.$$

To see this, we define

$$g(t) := \tilde{J}\left(w_0 + tv\right).$$

We know that $g$ has a unique minimum at $t = 0$. In particular:

$$\begin{aligned}
g(t) &= \tilde{J}\left(w_0 + tv\right) \\
&= \frac{1}{2}\int_\Omega |\nabla w_0|^2 \, \mathrm{d}x + t^2\frac{1}{2}\int_\Omega |\nabla v|^2 \, \mathrm{d}x + t\int_\Omega \nabla u_0 \cdot \nabla v \, \mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega |\nabla\varphi|^2 \, \mathrm{d}x + \int_\Omega \nabla w_0 \cdot \nabla\varphi \, \mathrm{d}x + t\int_\Omega \nabla v \cdot \nabla\varphi \, \mathrm{d}x \\
&\quad - \int_\Omega fu_0 \, \mathrm{d}x - t\int_\Omega fv \, \mathrm{d}x - \int_\Omega f\varphi \, \mathrm{d}x.
\end{aligned}$$

This is again a quadratic polynomial in $t$. The condition $g'(0) = 0$ ensures

$$\int_\Omega \nabla w_0 \cdot \nabla v \, \mathrm{d}x + \int_\Omega \nabla v \cdot \nabla\varphi \, \mathrm{d}x - \int_\Omega fv \, \mathrm{d}x = 0.$$

□

**4.3. Homogeous Neumann problem for $-\Delta + \mathrm{I}$.** We aim to prove that the weak solution of (3.1) is the unique minimum of a functional, which can be obtained using the Weierstrass Theorem. Consider the functional

$$J : H^1(\Omega) \to \mathbb{R}, \quad J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 \, \mathrm{d}x + \frac{1}{2}\int_\Omega |v|^2 \, \mathrm{d}x - \int_\Omega fv \, \mathrm{d}x.$$

We verify the hypotheses of the Weierstrass Theorem:

**Coercivity:** We estimate $J(v)$ from below by a quadratic polynomial in $\|v\|_{H^1(\Omega)}$:

$$J(v) = \frac{1}{2}\|v\|^2_{H^1(\Omega)} - \int_\Omega fv \, \mathrm{d}x \geqslant \frac{1}{2}\|v\|^2_{H^1(\Omega)} - \|f\|_2\|v\|_2 \geqslant \frac{1}{2}\|v\|^2_{H^1(\Omega)} - \|f\|_2\|v\|_{H^1(\Omega)}.$$

Thus, $J(v) \to +\infty$ as $\|v\|_{H^1(\Omega)} \to +\infty$.

**Strict Convexity:** Due to the linearity of the term $J_2$, we have

$$J\left(tv_1 + (1-t)v_2\right) - \left(tJ(v_1) + (1-t)J(v_2)\right) = J_1\left(tv_1 + (1-t)v_2\right) - \left(tJ_1(v_1) + (1-t)J_1(v_2)\right),$$

where $J_1$ is a squared norm, ensuring strict convexity.

**Lower Semicontinuity:** We rewrite the functional as

$$J(v) = \frac{1}{2}\|v\|^2_{H^1(\Omega)} - \int_\Omega fv \, \mathrm{d}x = J_1(v) + J_2(v).$$

The term $J_1$ is continuous and convex, hence weakly lower semicontinuous. The term $J_2$ is linear and continuous, thus weakly continuous. Therefore, $J$ is weakly lower semicontinuous.

By the Weierstrass Theorem, there exists a unique $u \in H^1(\Omega)$ that minimizes $J$. In particular, for any $t \in \mathbb{R}$ and any $v \in H^1(\Omega)$, we have $J(u) \leqslant J(u+tv)$.

We now show that $u$ is a weak solution of (3.1). Define $g(t) = J(u+tv)$. We know that $g$ has a unique minimum at $t=0$. Specifically:

$$\begin{aligned} g(t) = J(u+tv) &= \frac{1}{2}\int_\Omega |\nabla(u+tv)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |u+tv|^2\,\mathrm{d}x - \int_\Omega f(u+tv)\,\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \frac{t^2}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x + t\int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x \\ &\quad + \frac{1}{2}\int_\Omega |u|^2\,\mathrm{d}x + \frac{t^2}{2}\int_\Omega |v|^2\,\mathrm{d}x + t\int_\Omega uv\,\mathrm{d}x - \int_\Omega fu\,\mathrm{d}x - t\int_\Omega fv\,\mathrm{d}x. \end{aligned}$$

The condition $g'(0) = 0$ ensures

$$\int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x + \int_\Omega uv\,\mathrm{d}x - \int_\Omega fv\,\mathrm{d}x = 0.$$

**4.4. The Dirichlet eigenvalues of the Laplacian.** We consider the *Dirichlet eigenvalue problem* for the Laplace operator: Given a bounded open set $\Omega$, find $\lambda \in \mathbb{R}$ such that there exists a (non-trivial) weak solution $u$ to the problem

$$\begin{cases} -\Delta u = \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{4.4}$$

i.e., there exists $u \in H_0^1(\Omega)$ with $u \neq 0$, such that

$$\int_\Omega \nabla\varphi\cdot\nabla u = \lambda\int_\Omega u\varphi, \qquad \text{for all } \varphi \in H_0^1(\Omega).$$

The function $u$ is called *eigenfunction* associated with the eigenvalue $\lambda$.

REMARK 4.18. *The eigenvalue problem for the Laplacian arises, for example, when searching for solutions of the Schrödinger equation*

$$\mathrm{i}\partial_t u + \Delta u = 0$$

*that do not decay. Specifically, stationary solutions are sought, i.e., solutions of the form*

$$u(x,t) = \mathrm{e}^{\mathrm{i}\lambda t}w(x),$$

*with $\lambda > 0$. Substituting this ansatz into the equation yields a new equation:*

$$-\lambda w + \Delta w = 0.$$

THEOREM 4.25. *Every Dirichlet eigenvalue of $-\Delta$ in $\Omega$ is strictly positive and the associated eigenfunctions satisfy $\|u\|_{L^2} > 0$. Moreover, eigenfunctions associated to disinct eigenvalues are orthogonal with respect to the scalar product of $H_0^1(\Omega)$.*

PROOF. Let $\lambda$ be an eigenvalue, and let $u$ be an eigenfunction associated with $\lambda$. Taking $\varphi = u$ in the definition of a weak solution, we obtain

$$\int_\Omega |\nabla u|^2 = \lambda\int_\Omega |u|^2.$$

Since $u \neq 0$, the Poincaré inequality ensures that the first term is strictly positive, and thus the second term is also strictly positive. Since $\|u\|_2^2 \geqslant 0$, the claim follows.

Let $\lambda_1 \neq \lambda_2$ be distinct eigenvalues with corresponding eigenfunctions $u_1$ and $u_2$. By the definition of eigenfunctions, we have

$$\lambda_2\int_\Omega u_2u_1 = \int_\Omega \nabla u_1\cdot\nabla u_2 = \lambda_1\int_\Omega u_1u_2.$$

Subtracting side by side, we obtain

$$(\lambda_1 - \lambda_2)\int_\Omega u_2u_1 = 0,$$

which implies

$$\int_\Omega u_2u_1 = 0.$$

From this, we conclude the result, namely $\int_\Omega \nabla u_1 \cdot \nabla u_2 = 0$.

□

We are now interested in estimating the smallest eigenvalue from below. Let $C_P(n, \Omega, 2)$ be the Poincaré constant for $H_0^1(\Omega)$, namely,

$$\|u\|_2 \leqslant C_P(n, \Omega, 2)\|\nabla u\|_2 \qquad \text{for all } u \in H_0^1(\Omega).$$

Define $C(\Omega) = [C_P(n, \Omega, 2)]^{-2}$. It follows that

$$\int_\Omega |\nabla u|^2 \geqslant C(\Omega) \int_\Omega |u|^2 \qquad \text{for all } u \in H_0^1(\Omega).$$

This is equivalent to stating that

$$\frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} \geqslant C(\Omega) \qquad \text{for all } u \in H_0^1(\Omega),\ u \neq 0.$$

Let $\lambda$ be an eigenvalue with corresponding eigenfunction $u$. We have seen that $\|u\|_2^2 > 0$, and it satisfies

$$\lambda = \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} \geqslant C(\Omega) > 0.$$

Furthermore,

$$C(\Omega) \leqslant \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_2^2 = 1}} \int_\Omega |\nabla u|^2.$$

To prove the last equality, it suffices to substitute $u$ with the function $v = \frac{u}{\|u\|_2}$, which gives the relation $\leqslant$. The $\geqslant$ inequality is straightforward.

Our goal is to show that this infimum is attained by the smallest eigenvalue.

THEOREM 4.26. *Let*

$$\lambda_1 := \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^2(\Omega)} = 1}} \int_\Omega |\nabla u|^2.$$

*Then there exists* $\bar{u} \in H_0^1(\Omega)$, *with* $\|\bar{u}\|_{L^2(\Omega)} = 1$ *such that* $\lambda_1 = \int_\Omega |\nabla \bar{u}|^2$ *(i.e., the infimum is actually achieved),* $\lambda_1$ *is the smallest Dirichlet eigenvalue of* $-\Delta$ *in* $\Omega$.

PROOF. Let $\{u_n\}$ be a minimizing sequence for $\lambda_1$, with $u_n \in H_0^1(\Omega)$, $\|u_n\|_2 = 1$, and

$$\lim_{n \to \infty} \int_\Omega |\nabla u_n|^2 = \lambda_1.$$

In particular, $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is reflexive (as a Hilbert space), its bounded subsets are weakly relatively compact. Therefore, there exists a subsequence $\{u_{n_k}\}$ that converges weakly in $H_0^1(\Omega)$:

$$u_{n_k} \rightharpoonup \bar{u} \quad \text{with } \bar{u} \in H_0^1(\Omega).$$

Moreover, $u_{n_k} \rightharpoonup \bar{u}$ in $L^2(\Omega)$ and

$$\|\bar{u}\|_{H_0^1(\Omega)}^2 \leqslant \lim_{k \to \infty} \int_\Omega |\nabla u_{n_k}|^2 = \lambda_1.$$

We now apply Rellich-–Kondrachov's theorem for $p = q = 2$:

(i) If $n > 2$, then $H_0^1(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q \in \left[1, \frac{np}{n-p}\right)$;
(ii) If $n = 2$, then $H_0^1(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q \in [1, \infty)$;
(iii) If $n = 1$, then $H_0^1(\Omega) \hookrightarrow\hookrightarrow \mathcal{C}_0(\bar{\Omega})$.

For $n > 2$, $2 < \frac{2n}{n-2}$. For $n = 1$, since $\Omega$ is bounded, $\mathcal{C}_0(\bar{\Omega}) \hookrightarrow L^2(\Omega)$. In all cases,

$$H_0^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega).$$

Thus, from $\{u_{n_k}\}$, we can extract a subsequence that converges strongly in $L^2(\Omega)$. This new subsequence also converges to $\bar{u}$.

Using the continuity of the norm, we have

$$\|\bar{u}\|_{L^2(\Omega)} = 1.$$

Recalling that

$$\lambda_1 = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_2 = 1}} \int_\Omega |\nabla u|^2,$$

it follows that $\lambda_1 \leqslant \|\bar{u}\|^2_{H_0^1(\Omega)}$. From the weak convergence properties, we deduce $\lambda_1 = \|\bar{u}\|^2_{H_0^1(\Omega)}$, meaning $\lambda_1$ is a minimum, not just an infimum.

Step 2. We now show that $\lambda_1$ is an eigenvalue. Let $w \in H_0^1(\Omega)$ such that $\|w\|_2 = 1$ and $\int_\Omega |\nabla w|^2 = \lambda_1$. We prove that $w$ is the eigenfunction associated with $\lambda_1$.

Let $v \in H_0^1(\Omega)$ and $t \in \mathbb{R}$. By the definition of $\lambda_1$, we have

$$\lambda_1 \leqslant \frac{\int_\Omega |\nabla(w + tv)|^2}{\int_\Omega |w + tv|^2}.$$

Define the function $g : \mathbb{R} \to \mathbb{R}$ as

$$g(t) = \int_\Omega |\nabla(w + tv)|^2 - \lambda_1 \int_\Omega |w + tv|^2.$$

Then $g(t) \geqslant 0$ and $g(0) = 0$. Thus, $t = 0$ is the minimum of this function. It follows that $g'(0) = 0$. Calculating the derivative, we have

$$\begin{aligned} g(t) &= \int_\Omega |\nabla w|^2 + 2t \int_\Omega \nabla w \cdot \nabla v + t^2 \int_\Omega |\nabla v|^2 \\ &\quad - \lambda_1 \int_\Omega |w|^2 - 2\lambda_1 t \int_\Omega wv - \lambda_1 t^2 \int_\Omega |v|^2, \\ g'(t) &= 2 \int_\Omega \nabla w \cdot \nabla v + 2t \int_\Omega |\nabla v|^2 - 2\lambda_1 \int_\Omega wv - 2\lambda_1 t \int_\Omega |v|^2. \end{aligned}$$

The condition $g'(0) = 0$ is equivalent to

$$\int_\Omega \nabla w \cdot \nabla v - \lambda_1 \int_\Omega wv = 0,$$

showing that $w$ is the eigenfunction associated with $\lambda_1$.

It remains to prove that $\lambda_1$ is the smallest eigenvalue. Let $\lambda$ be another eigenvalue with associated eigenfunction $u_\lambda$. Using $u_\lambda$ in the weak solution formulation for the Dirichlet problem, we have

$$\int_\Omega |\nabla u_\lambda|^2 = \lambda \int_\Omega |u_\lambda|^2.$$

It follows that

$$\frac{\int_\Omega |\nabla u_\lambda|^2}{\int_\Omega |u_\lambda|^2} = \lambda \geqslant \lambda_1.$$

This concludes the proof.

□

PROPOSITION 4.3. *There exists a strictly increasing sequence of eigenvalues $\{\lambda_n\}$ with finite multiplicity such that $\lambda_n \to \infty$. Associated with this sequence is a corresponding sequence of eigenfunctions $\{u_n\}$, which forms a maximal orthonormal system in $H_0^1(\Omega)$.*

We do not present the proof, but we comment on the main idea. Given $\lambda_1, \ldots, \lambda_n$, one constructs

$$\lambda_{n+1} := \min\left\{\|\nabla u\|_2^2 : u \in E_n^\perp,\ \|u\|_2 = 1\right\},$$

where $E_n^\perp$ is the orthogonal complement of $E_n$ in $H_0^1(\Omega)$, and $E_n$ is generated by the eigenfunctions associated with $\lambda_1, \ldots, \lambda_n$.

EXAMPLE 4.11. *Let $\Omega = (0,1)$. The Dirichlet problem is given by*

$$\begin{cases} u'' + \lambda u = 0, \\ u(0) = u(1) = 0, \end{cases} \qquad \text{in } (0,1).$$

*Since $u(0) = 0$, the solution takes the form $u(x) = c\sin(\sqrt{\lambda}x)$. By choosing $c$ appropriately depending on $\lambda$, we ensure $\|u\|_2 = 1$.*

*Imposing* $u(1) = 0$, *and since* $u \neq 0$, *we find* $\sin(\sqrt{\lambda}) = 0$, *which implies* $\lambda = \pi^2 n^2$ *for* $n \in \mathbb{N}\backslash\{0\}$.

*The eigenvalues are all simple, and the eigenfunctions form a maximal orthonormal system in* $H_0^1(0,1)$ *(as follows from the theory of Fourier series).*

4.4.1. *A nonlinear eigenvalue problem.* Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, with $n \geqslant 3$, and let $2 \leqslant p < 2^* = \frac{2n}{n-2}$. We aim to show that

$$M_1 := \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^p\right)^{2/p}}$$

is a minimum and to determine its value. For $p > 2$, the problem is nonlinear, whereas for $p = 2$, the problem becomes linear and will be revisited in the last section.

Recall that since $\Omega$ is bounded, the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ is continuous for every $1 \leqslant p \leqslant 2^*$. In particular, $u \in H_0^1(\Omega)$ has a finite $L^p(\Omega)$-norm. By homogeneity, we can reduce the problem to finding

$$M := \inf_{u \in E} \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x,$$

where

$$E := H_0^1(\Omega) \cap \{u \in L^p(\Omega) : \|u\|_p = 1\}.$$

To see the role of homogeneity, observe that if $u \in E$, then

$$\int_\Omega |\nabla u|^2 \, \mathrm{d}x = \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}x}{\|u\|_p^2} \geqslant M_1.$$

Taking the infimum, we obtain $M \geqslant M_1$. Conversely, if $u \in H_0^1(\Omega), u \neq 0$, then $\frac{u}{\|u\|_p} \in E$, and thus

$$\frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}x}{\|u\|_p^2} = \int_\Omega \left|\nabla \frac{u}{\|u\|_p}\right|^2 \mathrm{d}x \geqslant M,$$

which implies $M_1 \geqslant M$. Therefore, $M_1 = M$.

PROPOSITION 4.4. *$M$ is finite and is a minimum.*

PROOF. Since the functional $u \mapsto \|\nabla u\|_2^2$ is positive, $M$ is finite. Let $\{u_n\}$ be a minimizing sequence for $M$, with $u_n \in H_0^1(\Omega)$, $\|u_n\|_p = 1$, and

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 = M.$$

In particular, $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is reflexive (being a Hilbert space), its bounded subsets are weakly relatively compact. Thus, there exists a subsequence $\{u_{n_k}\}$ that converges weakly in $H_0^1(\Omega)$:

$$u_{n_k} \rightharpoonup \bar{u} \quad \text{with } \bar{u} \in H_0^1(\Omega).$$

This implies $\nabla u_{n_k} \rightharpoonup \nabla \bar{u}$ in $L^2(\Omega)$ and $u_{n_k} \rightharpoonup \bar{u}$ in $L^p(\Omega)$. By the norm convergence property,

$$\|\bar{u}\|_{H_0^1(\Omega)}^2 \leqslant \liminf_{k\to\infty} \int_\Omega |\nabla u_{n_k}|^2 = M.$$

We now apply Rellich—Kondrachov's theorem (here, the assumption $p < 2^*$ is crucial):

$$H_0^1(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega).$$

Thus, from $\{u_{n_k}\}$, we extract a subsequence that converges strongly in $L^p(\Omega)$. Since $\Omega$ is bounded and $p \geqslant 2$, this subsequence also converges weakly in $L^2(\Omega)$, and by the uniqueness of the weak limit, its limit is $\bar{u}$. Using the continuity of the norm, we have $\|\bar{u}\|_{L^p(\Omega)} = 1$, and hence $\bar{u} \in E$. By the definition of $M$, we have $M \leqslant \|\bar{u}\|_{H_0^1(\Omega)}^2$, but from the weak convergence properties, we deduce $M = \|\bar{u}\|_{H_0^1(\Omega)}^2$. Thus, $M$ is a minimum, not just an infimum. $\square$

PROPOSITION 4.5. *Consider the nonlinear problem for $2 < p < 2^*$ on $\Omega$ bounded:*

$$\begin{cases} -\Delta u = M|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

*It admits a weak solution $\bar{u} \in H_0^1(\Omega)$ such that $\|\bar{u}\|_p = 1$ and*

$$\int_\Omega \nabla\bar{u} \cdot \nabla v - M \int_\Omega |\bar{u}|^{p-2}\bar{u}v = 0 \qquad \text{for all } v \in H_0^1(\Omega).$$

*Moreover,*

$$M = \int_\Omega |\nabla\bar{u}|^2 = \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^p\right)^{2/p}}.$$

PROOF. Let $\bar{u} \in H_0^1(\Omega) \cap \{u \in L^p(\Omega) : \|u\|_p = 1\}$ be such that

$$M := \int_\Omega |\nabla\bar{u}(x)|^2 \,\mathrm{d}x.$$

We now show that $\bar{u}$ satisfies the following Euler–Lagrange equation (in weak form):

$$\int_\Omega \nabla\bar{u} \cdot \nabla v - M \int_\Omega |\bar{u}|^{p-2}\bar{u}v = 0 \qquad \text{for all } v \in H_0^1(\Omega).$$

Let $v \in H_0^1(\Omega)$ and $t \in \mathbb{R}$. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) = \int_\Omega |\nabla(\bar{u} + tv)|^2 - M\left(\int_\Omega |\bar{u} + tv|^p\right)^{2/p}.$$

By the definition of $M$ and $\bar{u}$, we have $g(t) \geqslant 0$, $g(0) = 0$, and $g$ has a minimum at $t = 0$. Therefore, $g'(0) = 0$. Differentiating and using the parameter-dependent integral theory, we find

$$\begin{aligned} g'(t) &= 2\int_\Omega \nabla(\bar{u} + tv) \cdot \nabla v - M\frac{2}{p}\left(\int_\Omega |\bar{u} + tv|^p\right)^{2/p-1} \int_\Omega p|\bar{u} + tv|^{p-2}(\bar{u} + tv)v, \\ g'(0) &= 2\int_\Omega \nabla\bar{u} \cdot \nabla v - 2M\int_\Omega |\bar{u}|^{p-2}\bar{u}v. \end{aligned}$$

The condition $g'(0) = 0$ is then equivalent to the weak form of the equation. □

REMARK 4.19. *Note that the previous proof does not guarantee uniqueness of the solution.*

REMARK 4.20. *Is it necessary for the constant in front of the nonlinear term to be $M$, the minimum we started from? Clearly not. Let $\mu > 0$, and substitute $u(x)$ with $\mu u(x)$ in the previous problem. We have*

$$\begin{cases} -\mu\Delta v = M\mu^{p-1}|v|^{p-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

*which implies*

$$\begin{cases} -\Delta v = M\mu^{p-2}|v|^{p-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

*For $\lambda > 0$, setting $\mu = \lambda^{1/(p-2)}M^{-1/(p-2)}$, $v$ solves*

$$\begin{cases} -\Delta v = \lambda|v|^{p-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

REMARK 4.21. *We cannot consider $\lambda < 0$, because choosing $u = v$ in the weak solution definition would yield*

$$\int |\nabla u|^2 = \lambda \int |u|^p,$$

*which forces $u = 0$, violating the condition $\|u\|_p = 1$.*

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**4.5. A semilinear problem.** We consider the semilinear problem

$$\begin{cases} -\Delta u = |u|^{p-1}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{4.5}$$

THEOREM 4.27. *If $n \geqslant 3$ and $1 < p < \frac{n+2}{n-2}$, then there exists a non-trivial weak solution $u \in H_0^1(\Omega)$ of (4.5).*

PROOF. We define $F, G : H_0^1(\Omega) \to \mathbb{R}$ by

$$F(u) = \int_\Omega |Du|^2 \, \mathrm{d}x, \quad G(u) = \int_\Omega |u|^{p+1} \, \mathrm{d}x - 1$$

and the admissible set

$$\mathcal{A} := \left\{ u \in H_0^1(\Omega) : G(u) = 0 \right\}.$$

Step 1. We show that $G$ is well-defined on $H_0^1(\Omega)$. By Sobolev's embedding theorem,

$$H_0^1(\Omega) \subset L^{p+1}(\Omega) \qquad \text{forall } 1 \leqslant p \leqslant \frac{2n}{n-2} - 1 = \frac{n+2}{n-2},$$

so $G$ is well defined on $H_0^1(\Omega)$ for all $1 < p < \frac{n+2}{n-2}$.

Step 2. Now that we know the problem is well-defined, we note that $F$ is Gâteaux differentiable on $H_0^1(\Omega)$ with

$$dF[u]v = 2 \int_\Omega Du \cdot Dv \, \mathrm{d}x \qquad \text{for all } u, v \in H_0^1(\Omega)$$

and that $G$ is $C^1$ on $H_0^1(\Omega)$ with[14]

$$DG(u)v = (p+1) \int_\Omega |u|^{p-1} uv \, \mathrm{d}x, \qquad \text{for all } u, v \in H_0^1(\Omega).$$

Step 3. The Lagrange multiplier theorem implies that if $\phi \in \mathcal{A}$ is a minimizer of $F|_\mathcal{A}$, then there exists a $\lambda \in \mathbb{R}$ such that

$$2 \int_\Omega D\phi \cdot Dv dx = \lambda(p+1) \int_\Omega |\phi|^{p-1} \phi v dx.$$

for all $v \in H_0^1(\Omega)$, that is, $\phi$ is a non-trivial weak solution of

$$\begin{cases} -\Delta u = \mu |u|^{p-1}u, & x \in \Omega, \\ u = 0, x \in \partial\Omega, \end{cases}$$

where $\mu = \frac{\lambda(p+1)}{2}$. Note that, taking $v = \phi$, gives, in particular

$$2 \int_\Omega |D\phi|^2 dx = \lambda(p+1) \int_\Omega |\phi|^{p+1} dx$$

so that $\lambda \geqslant 0$. However, note that if $\lambda = 0$ then $\int_\Omega |D\phi|^2 dx = 0$ so that $\phi$ is a constant in $H_0^1(\Omega)$, i.e., $\phi = 0$ a.e. in $\Omega$, which is a contradiction. Now rescaling $\phi$ as

$$w = \mu^{1/(p-1)} \phi$$

it follows that $w$ is a non-trivial weak solution of (4.5), as desired.

Step 4. In summary, the existence of non-trivial weak solutions of (4.5) can be established by showing that the problem

$$\lambda = \inf_{u \in \mathcal{A}} F(u)$$

has a minimizer in $\mathcal{A}$. To apply Theorem 4.23 to show that $F|_\mathcal{A}$ has a minimizer, we observe that

(1) The functional $F$ is clearly bounded below on $H_0^1(\Omega)$.

---

[14] By Taylor's theorem,

$$|u + \epsilon v|^{p+1} = |u|^{p+1} + (p+1)\epsilon |u|^{p-1} uv + \frac{p(p+1)}{2} \epsilon^2 |u + \theta v|^{p-1} v^2,$$

for some $0 < \theta(x) < \epsilon$.

(2) $F$ is coercive on $H_0^1(\Omega)$. Indeed, by Poincaré we have

$$\int_\Omega u^2 dx \leqslant C \int_\Omega |Du|^2 dx = CF(u)$$

for some constant $C > 0$, and hence $F(u) \to \infty$ as $\|u\|_{H^1(\Omega)} \to \infty$.

(3) $F$ is weakly lower semicontinuous on $H_0^1(\Omega)$.

It remains to verify that $G : H_0^1(\Omega) \to \mathbb{R}$ is continuous with respect to weak convergence. This follows because Rellich–Kondrachov's theorem with $p = 2$ and $n \geqslant 3$ implies that

$$H_0^1(\Omega) \Subset L^{p+1}(\Omega) \text{ for all } 0 \leqslant p < \frac{2n}{n-2} - 1 = \frac{n+2}{n-2}.$$

Since $G(u) = \int_\Omega |u|^{p+1} dx - 1$, we have that $G$ is continuous with respect to weak convergence in $H_0^1(\Omega)$. In summary, by Theorem 4.23, if $1 < p < \frac{n+2}{n-2}$ then there exists a $\phi \in \mathcal{A}$ such that

$$F(\phi) = \min_{u \in \mathcal{A}} F(u)$$

and hence $\phi$ induces a non-trivial weak solution of (4.5). □

**4.6. Pohozaev's identity and a non-existence result for a supercritical semilinear problem.** We conclude this chapter by proving a non-existence result for a semilinear elliptic problem, with supercritical growth, in star-shaped domains.[15]

THEOREM 4.28. *Let $\Omega \subset \mathbb{R}^n$ be an open star-shaped set with $C^1$ boundary, with $n \geqslant 3$. If $u \in C^2(\bar{\Omega})$ is a classical solution of*

$$\begin{cases} -\Delta u = |u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{4.6}$$

*for some $p > 2^* = 2n/(n-2)$, then $u \equiv 0$ in $\Omega$.*

This result is a consequence of Derrick–Pohozaev's identity.[16] An often used more general form is due to Henri Berestycki and Pierre-Louis Lions [BL83]. The proof of Derrick–Pohozaev's identity is a remarkable calculation initiated by multiplying the $-\Delta u = |u|^{p-1}u$ by $x \cdot Du$ and integrating by parts.

LEMMA 4.9 (Derrick–Pohozaev's identity). *Let $u$ be a solution of (4.6). Then*

$$\frac{n-2}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \nu(x) \cdot x \, \mathrm{d}S(x) = \frac{n}{p} \int_\Omega |u|^p \, \mathrm{d}x,$$

*where $\nu(x)$ denotes the outward-pointing normal vector to $x$.*

PROOF. We multiply the PDE by $x \cdot Du$ and integrate over $\Omega$, to find

$$\underbrace{\int_\Omega (-\Delta u)(x \cdot Du) \, \mathrm{d}x}_{=:A} = \underbrace{\int_\Omega |u|^{p-2} u (x \cdot Du) \, \mathrm{d}x}_{=:B}$$

<u>Step 1.</u> The term on the left is

$$\begin{aligned} A &= -\sum_{i,j=1}^n \int_\Omega u_{x_i x_i} x_j u_{x_j} dx \\ &= \sum_{i,j=1}^n \int_\Omega u_{x_i} \left(x_j u_{x_j}\right)_{x_i} dx - \sum_{i,j=1}^n \int_{\partial\Omega} u_{x_i} \nu^i x_j u_{x_j} dS \\ &=: A_1 + A_2. \end{aligned}$$

[15] An open set $\Omega$ is called *star-shaped with respect to* 0 if, for each $x \in \bar{\Omega}$, the line segment $\{\lambda x : 0 \leqslant \lambda \leqslant 1\}$ lies in $\bar{\Omega}$. If $\Omega$ is convex and $0 \in \Omega$, then $\Omega$ is star-shaped with respect to 0, but a general star-shaped region need not be convex.

[16] Named after S. I. Pohožaev [Poh65] and G. H. Derrick [Der64].

For $A_1$, we compute

$$\begin{aligned} A_1 &= \sum_{i,j=1}^{n} \int_\Omega u_{x_i}\delta_{ij}u_{x_j} + u_{x_i}x_j u_{x_j x_i} dx \\ &= \int_\Omega |Du|^2 + \sum_{j=1}^{n} \left(\frac{|Du|^2}{2}\right)_{x_j} x_j dx \\ &= \left(1-\frac{n}{2}\right)\int_\Omega |Du|^2 dx + \int_{\partial\Omega} \frac{|Du|^2}{2}(\nu\cdot x)\, dS \end{aligned}$$

Here we really used

$$\Delta u = \nabla\cdot(\nabla u) = \nabla\cdot f$$

and

$$\nabla(x\cdot\nabla u) = \nabla u + x\cdot\nabla^2 u.$$

On the other hand, since $u = 0$ on $\partial\Omega$, $Du(x)$ is parallel to the normal $\nu(x)$ at each point $x\in\partial\Omega$. Thus, $Du(x) = \pm|Du(x)|\nu(x)$. Using this equality, we calculate

$$A_2 = -\int_{\partial\Omega} |Du|^2(\nu\cdot x)\,\mathrm{d}S.$$

In summary, we deduce

$$A = \frac{2-n}{2}\int_\Omega |Du|^2 dx - \frac{1}{2}\int_{\partial\Omega}|Du|^2(\nu\cdot x)\,\mathrm{d}S.$$

<u>Step 2.</u> For $B$, we compute

$$\begin{aligned} B &:= \sum_{j=1}^{n}\int_\Omega |u|^{p-2}u x_j u_{x_j}\,\mathrm{d}x \\ &= \sum_{j=1}^{n}\int_\Omega \left(\frac{|u|^p}{p}\right)_{x_j} x_j\,\mathrm{d}x = -\frac{n}{p+1}\int_\Omega |u|^p\,\mathrm{d}x. \end{aligned}$$

<u>Step 3.</u> Putting Step 1 and Step 2 together, we conclude

$$\left(\frac{n-2}{2}\right)\int_\Omega |Du|^2\,\mathrm{d}x + \frac{1}{2}\int_{\partial\Omega}|Du|^2(\nu\cdot x)\,\mathrm{d}S = \frac{n}{p}\int_\Omega |u|^p\,\mathrm{d}x.$$

□

We also need a lemma on star-shaped sets.

Lemma 4.10 (Normals to a star-shaped region). *Let $\Omega$ be an open star-shaped set in $\mathbb{R}^n$ with $C^1$ boundary. Then $x\cdot\nu(x)\geqslant 0$ for all $x\in\partial\Omega$ (where $\nu(x)$ denotes the unit outward normal to $\partial\Omega$ at $x$).*

Proof. Since $\partial\Omega$ is $C^1$, if $x\in\partial\Omega$ then for each $\epsilon>0$ there exists $\delta>0$ such that $|y-x|<\delta$ and $y\in\bar{U}$ imply $\nu(x)\cdot\frac{(y-x)}{|y-x|}\leqslant\epsilon$.[17] In particular

$$\limsup_{\substack{y\to x\\ y\in\bar\Omega}} \nu(x)\cdot\frac{(y-x)}{|y-x|}\leqslant 0.$$

Let $y=\lambda x$ for $0<\lambda<1$. Then $y\in\bar\Omega$, since $\Omega$ is star-shaped. Thus, noticing that

$$\frac{\lambda x - x}{|\lambda x - x|} = \frac{(\lambda-1)x}{|\lambda-1||x|} = -\frac{x}{|x|}$$

as $\lambda\in(0,1)$, we deduce

$$\nu(x)\cdot\frac{x}{|x|} = -\lim_{\lambda\to 1^-}\nu(x)\cdot\frac{(\lambda x - x)}{|\lambda x - x|}\geqslant 0.$$

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[17]We write $\Omega = F^{-1}(-\infty,0)$ and $\partial\Omega = F^{-1}(0)$ and $\nu = \nabla F$ for a $C^1$ function $F$. We are then claiming that, for any $\epsilon>0$, there is a $\delta>0$ such that if $|y-x|<\delta$ and $F(y)\leqslant F(x)$, then

$$\nabla F(x)\cdot\frac{y-x}{|y-x|}<\epsilon.$$

□

PROOF OF THEOREM 4.28. Using Lemma 4.9 and Lemma 4.10, we have

$$\frac{n-2}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x \leqslant \frac{n}{p}\int_\Omega |u|^p \,\mathrm{d}x.$$

But, since we assumed that $u$ is a (classical) solution, we have

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x = \int_\Omega |u|^p \,\mathrm{d}x.$$

Combining these two observations implies

$$\left(\frac{n-2}{2} - \frac{n}{p}\right)\int_\Omega |u|^p \,\mathrm{d}x \leqslant 0.$$

Hence, if $u \not\equiv 0$, it follows that $\frac{n-2}{2} - \frac{n}{p} \leqslant 0$, that is, $p \leqslant \frac{2n}{n-2}$. This yields $u \equiv 0$ if $p > 2^*$. □

CHAPTER 5

# General second-order elliptic PDEs

## 1. Maximum principles for uniformly elliptic operators

**Lect. 11, 26.11**

In this section, we revisit the weak and strong maximum principles established earlier for harmonic functions. Specifically, we will apply the methods outlined in Section 2.3.3 and Section 2.3.6 to extend these results to uniformly elliptic operators of the form

$$Lu(x) := -\sum_{i,j=1}^{n} a_{ij}(x)\partial^2_{x_i x_j}u(x) + \sum_{i=1}^{n} b_i(x)\partial_{x_i}u(x) + c(x)u(x), \tag{1.1}$$

acting on functions $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is an open set. Let us introduce the matrix valued function $x \mapsto A(x) = (a_{ij}(x))_{i,j=1}^n$ and the vector valued function $x \mapsto b(x) = (b_i(x))_{i=1}^n$ so that $L$ can be written in more compact form as

$$Lu(x) = -A(x) : D^2u(x) + b(x)\cdot Du(x) + c(x)u(x),$$

where $D^2u$ denotes the Hessian of $u$, $Du$ its gradient and, for matrices $A, B \in \mathbb{R}^{n\times k}$, we have used the notation $A : B := \sum_{i=1}^n \sum_{j=1}^k A_{ij}B_{ij}$.

We say that the operator $L$ is *elliptic* if the matrix $(a_{ij})$ is positive definite (this is consistent with Definition 1.6).

Definition 5.1 (Uniformly elliptic operator). *The operator $L$ in (1.1) is said to be* uniformly elliptic *in $\Omega$ if there exist $\lambda, \Lambda \in \mathbb{R}_+$ such that*

$$0 < \lambda|\xi|^2 \leqslant \xi^\top A(x)\xi \leqslant \Lambda|\xi|^2, \qquad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n. \tag{1.2}$$

In what follows, will also assume that there exist constants $B$, $C > 0$ such that

$$|b(x)| \leqslant B, \quad 0 \leqslant c(x) \leqslant C, \qquad \text{for all } x \in \Omega. \tag{1.3}$$

Example 5.1. *The* Laplacian operator *$L = -\Delta$ is uniformly elliptic on any open set, with $\theta = 1$.*

Example 5.2. *The* Tricomi operator

$$L = y\partial_x^2 + \partial_y^2$$

*is elliptic in $y > 0$ and hyperbolic in $y < 0$. For any $0 < \epsilon < 1, L$ is uniformly elliptic in the strip $\{(x, y) : \epsilon < y < 1\}$, with $\theta = \epsilon$, but it is not uniformly elliptic in $\{(x, y) : 0 < y < 1\}$.*

Remark 5.1. *We recall a linear algebra result that will be helpful going forward. Let $A \in \mathbb{R}^{n\times n}$ be a positive definite* matrix *(i.e., $\xi^\top A\xi > 0$ for all $\neq \xi \in \mathbb{R}^n$) and $D \in \mathbb{R}^{n\times n}$ be a positive semi-definite symmetric matrix. Then*

$$A : D = \sum_{i,j} A_{ij}D_{ij} \geqslant 0.$$

*Indeed, $D$ can be diagonalized as $D = \sum_{l=1}^n \lambda_l v^{(l)}\left(v^{(l)}\right)^\top$ where $\lambda_l \geqslant 0$ are the (real) eigenvalues of $D$ and $v^{(l)} \in \mathbb{R}^n$ the corresponding eigenvectors. Then*

$$A : D = \sum_{l=1}^{n} \lambda_l \sum_{i,j=1}^{n} v_i^{(l)} A_{ij} v_j^{(l)} = \sum_{l=1}^{n} \lambda_l \left(v^{(l)}\right)^\top A v^{(l)} \geqslant 0.$$

**1.1. Weak maximum principle.** As a first result, we generalize the weak maximum principl to the case of an elliptic operator on a bounded domain. We start from the easier case $c = 0$.

THEOREM 5.1 (Weak maximum principle for uniformly elliptic operators (case $c = 0$)). *Let $\Omega$ be a bounded domain, $L$ a uniformly elliptic operator in $\Omega$ and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$.*

*(1) If $Lu \leqslant 0$ and $c = 0$ in $\Omega$, then $\max_{x\in\bar{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x)$.*
*(2) If $Lu \geqslant 0$ and $c = 0$ in $\Omega$, then $\min_{x\in\bar{\Omega}} u(x) = \min_{x\in\partial\Omega} u(x)$.*

PROOF. We will prove only (1) (while the proof of (2) is analogous).
Step 1: $Lu < 0$. We consider first the case $Lu < 0$ and prove the statement by contradiction. Suppose that $\max_{x\in\bar{\Omega}} u(x) > \max_{x\in\partial\Omega} u(x)$. Then, there exists $x_0 \in \Omega$ such that $u(x_0) = \max_{x\in\bar{\Omega}} u(x)$ and, since $u \in C^2(\Omega)$, we have that $-D^2u(x_0)$ is a symmetric and positive semi-definite matrix and $Du(x_0) = 0$. This implies (recalling Remark 5.1)

$$Lu(x_0) = \underbrace{-A(x_0) : D^2u(x_0)}_{\geqslant 0} + \underbrace{b(x_0) \cdot Du(x_0)}_{=0} \geqslant 0,$$

which contradicts the assumption.
Step 2: $Lu \leqslant 0$. Consider now the case $Lu \leqslant 0$ and the auxiliary (comparison) function $\varphi(x) := e^{\gamma x_1}$ with $\gamma > \frac{B}{\lambda}$ (where $B$ is the constant in (1.3)). Since $a_{11}(x) = e_1^\top A(x)e_1 \geqslant \lambda$, we have

$$L\varphi(x) = -\gamma^2 a_{11}(x)e^{\gamma x_1} + \gamma b_1(x)e^{\gamma x_1} \leqslant \gamma(-\gamma\lambda + B)e^{\gamma x_1} < 0.$$

Then, for any $\epsilon > 0$, we define $v_\epsilon(x) := u(x) + \epsilon\varphi(x)$, which satisfies $Lv_\epsilon < 0$ in $\Omega$. Hence, by Step 1, $\max_{\bar{\Omega}} v_\epsilon = \max_{\partial\Omega} v_\epsilon$. Letting $\epsilon \to 0$, this yields $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$. □

In the case $c \geqslant 0$, the result has to be slightly weakened.

THEOREM 5.2 (Weak maximum principle for uniformly elliptic operators (case $c \geqslant 0$)). *Let $\Omega$ be a bounded domain, $L$ a uniformly elliptic operator in $\Omega$ and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$.*

*(1) If $c \geqslant 0$ and $Lu \leqslant 0$ in $\Omega$ then*

$$\max_{x\in\bar{\Omega}} u(x) \leqslant \max_{x\in\partial\Omega} u^+(x),$$

*where $u^+ := \max\{u, 0\}$.*

*(2) If $c \geqslant 0$ and $Lu \geqslant 0$ in $\Omega$ then*

$$\min_{x\in\bar{\Omega}} u(x) \geqslant \min_{x\in\partial\Omega} \left(-u^-(x)\right),$$

*where $u^- := -\min\{u, 0\}$.*

*(3) In particular, if $Lu = 0$ in $\Omega$, then $\max_{x\in\bar{\Omega}} |u(x)| = \max_{x\in\partial\Omega} |u(x)|$.*

PROOF. We prove (1) (as the proof of (2) is analogous).
Let $\Omega_+ := \{x \in \bar{\Omega} : u(x) > 0\}$ and $L_0u = -A : D^2u + b \cdot Du$. If $\Omega_+ = \varnothing$, the result is true. Otherwise, in $\Omega_+$, we have

$$L_0u = Lu - \underbrace{cu}_{\geqslant 0} \leqslant 0,$$

so, by Theorem 5.1, $M := \max_{x\in\bar{\Omega}_+} u(x) = \max_{x\in\partial\Omega_+} u(x)$ and $M \geqslant 0$.

On the one hand, $\max^+_{x\in\bar{\Omega}} u(x) = \max_{x\in\bar{\Omega}_+} u(x)$. On the other hand, let $x_0 \in \partial\Omega_+$ be such that $u(x_0) = M$. If $x_0 \in \Omega$, then $M = 0$ (by the continuity of $u$) and if $x_0 \in \partial\Omega$, then $M = u(x_0) = \max_{\partial\Omega} u$. Hence,

$$\max_{x\in\bar{\Omega}} u(x) = \max_{x\in\partial\Omega} u^+(x).$$

Finally, to prove (3), we note that, if $Lu = 0$ in $\Omega$ then, in particular,

$$\begin{array}{lll} Lu \leqslant 0 & \text{and} & u(y) \leqslant \max_{x\in\partial\Omega} u^+(x) \leqslant \max_{x\in\partial\Omega} |u(x)| \\ Lu \geqslant 0 & \text{and} & u(y) \geqslant \min_{x\in\partial\Omega} (-u^-(x)) \geqslant -\max_{x\in\partial\Omega} |u(x)|. \end{array}$$

Hence, $|u(y)| \leqslant \max_{x\in\partial\Omega} |u(x)|$ for all $y \in \Omega$, which yields (3). □

From Theorem 5.2, we can deduce a comparison/positivity principle.

COROLLARY 5.1 (Comparison principle). *Let $L$ be uniformly elliptic and $c \geqslant 0$ in $\Omega$.*

*(1) If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $Lu \leqslant 0$ in $\Omega, u \leqslant 0$ on $\partial\Omega$, then $u \leqslant 0$ in $\bar{\Omega}$.*
*(2) If $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy $Lu \leqslant Lv$ in $\Omega, u \leqslant v$ on $\partial\Omega$, then $u \leqslant v$ in $\bar{\Omega}$.*

1.1.1. *A priori bound on the solutions of the Dirichlet problem.* The maximum principle allows us also to derive a simple a priori bound on the solutions of the Dirichlet problem

$$\begin{cases} Lu = f, & x \in \Omega, \\ u = g, & x \in \partial\Omega. \end{cases} \tag{1.4}$$

In turn, this bound immediately gives uniqueness of solutions to the Dirichlet problem (1.4).

PROPOSITION 5.1 (A priori bounds for the Dirichlet problem). *Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be a solution of the Dirichlet problem (1.4), with $\Omega$ bounded, $L$ uniformly elliptic in $\Omega$ with $c \geqslant 0$, $f \in C^{0,\alpha}(\Omega)$ for some $\alpha \in (0,1)$ and bounded, $g \in C^0(\partial\Omega)$. Then*

$$\|u\|_{C^0(\bar{\Omega})} \leqslant \|g\|_{C^0(\partial\Omega)} + C \sup_{x\in\Omega} |f(x)|,$$

*with $C = \frac{2}{\lambda} e^{2\left(\frac{B}{\lambda}+1\right)D}$, where $\lambda, B$ are as in (1.2)–(1.3), and $D := \sup_{x\in\Omega} |x|$.*

PROOF. Let $L_0 u := -A : D^2 u + b \cdot Du$ and take a comparison function $\tilde{\varphi}$ that satisfies

$$L_0 \tilde{\varphi} \leqslant -\alpha, \quad |\tilde{\varphi}| \leqslant \beta \quad \text{in } \Omega,$$

for some $\alpha, \beta > 0$. For instance, we can take the comparison function $\tilde{\varphi}(x) := e^{\gamma x_1}$ considered in the proof of Theorem 5.1, for which we have already established the bound $L_0\tilde{\varphi}(x) \leqslant \gamma(-\gamma\lambda + B)e^{\gamma x_1}$. Therefore, if we take $\gamma = 1 + \frac{B}{\lambda}$, we have

$$L_0\tilde{\varphi}(x) \leqslant -\gamma\lambda e^{\gamma x_1} \leqslant -\lambda e^{-\gamma D} =: -\alpha, \quad |\tilde{\varphi}(x)| \leqslant e^{\gamma D} =: \beta, \qquad \text{for all } x \in \Omega$$

Let us now set $G := \|g\|_{C^0(\partial\Omega)}$, $F = \sup_\Omega |f|$, and consider another comparison function

$$\varphi(x) := G + (\beta - \tilde{\varphi}(x))\frac{F}{\alpha},$$

which satisfies $\varphi \geqslant 0$ in $\Omega$, $\varphi \geqslant g$ on $\partial\Omega$, and

$$L\varphi = L_0\varphi + c\varphi = -L_0\tilde{\varphi}\frac{F}{\alpha} + \underbrace{c\varphi}_{\geqslant 0} \geqslant F.$$

Hence,

$$L(\varphi - u) \geqslant F - f(x) \geqslant 0, \quad \varphi - u \geqslant 0 \qquad \text{on } \partial\Omega$$

and, by the weak maximum principle, $\varphi - u \geqslant 0$ in $\bar{\Omega}$. This yields

$$u(x) \leqslant G + \frac{2\beta}{\alpha} F.$$

The lower bound can be obtained by taking $-\varphi$ instead of $\varphi$. □

**1.2. Strong maximum principle.** The strong maximum principle holds as well for a uniformly elliptic operator on a domain $\Omega$.

THEOREM 5.3 (Strong maximum principle). *Let $L$ be uniformly elliptic in $\Omega$ (with $\Omega$ not necessarily bounded) and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$.*

*(1) If $c = 0$ and $Lu \leqslant 0$ (resp., $Lu \geqslant 0$) in $\Omega$ and there exists $y \in \Omega$ such that $u(y) = \max_{x\in\bar{\Omega}} u(x)$ (resp., $u(y) = \min_{x\in\bar{\Omega}} u(x)$), then $u$ is constant in $\bar{\Omega}$.*

*(2) If $c \geqslant 0$ and $Lu \leqslant 0$ (resp., $Lu \geqslant 0$) in $\Omega$ and there exists $y \in \Omega$ such that $u(y) = \max_{x\in\bar{\Omega}} u(x)$ and $u(y) \geqslant 0$ (resp., $u(y) = \min_{x\in\bar{\Omega}} u(x)$ and $u(y) \leqslant 0$), then $u$ is constant in $\bar{\Omega}$.*

For the proof of Theorem 5.3, we need first to establish, as an auxiliary result, a generalization of Lemma 2.1.

LEMMA 5.1 (Zaremba–Hopf–Oleĭnik's boundary point lemma). *Let $L$ be uniformly elliptic in $\Omega$, $c \geqslant 0$, and $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that $Lu \leqslant 0$ and $\max_{\partial\Omega} u \geqslant 0$ if $c \neq 0$. If there exists $x_0 \in \partial\Omega$ maximizing $u$ in $\bar{\Omega}$ such that*

*(1) $u(x_0) > u(x)$ for all $x \in \Omega$,*

*(2) $\partial\Omega$ satisfies an interior sphere condition at $x_0$, i.e., there exists $y \in \Omega$ and $r > 0$ such that $B_r(y) \subset \Omega$ and $\overline{B_r(y)} \cap \partial\Omega = \{x_0\}$,*

*then $\partial_\nu u(x_0) > 0$ (with strict inequality).*

PROOF. Let $\rho \in (0,r)$ and consider the annular region $\Sigma := \{x \in B_r(y) : 0 < \rho < |x-y| < r\}$, and $\bar{\Sigma} \cap \partial\Omega = \{x_0\}$.

We define the auxiliary function $\varphi(x) := e^{-\alpha|x-y|^2} - e^{-\alpha r^2}$, which satisfies $\varphi \geqslant 0$ in $\Sigma$ and

$$\begin{aligned} L\varphi(x) &= e^{-\alpha|x-y|^2}\left[-\sum_{i,j} a_{ij}4\alpha^2\left(x_i-y_i\right)\left(x_j-y_j\right) + \sum_i 2\alpha\left(a_{ii} - b_i\left(x_i-y_i\right)\right) + c\right] - ce^{-\alpha r^2} \\ &\leqslant e^{-\alpha|x-y|^2}\left[-4\alpha^2\lambda|x-y|^2 + 2\alpha(n\Lambda + |b||x-y|) + c\right] \\ &\leqslant e^{-\alpha|x-y|^2}\left[-4\alpha^2\rho^2\lambda + 2\alpha(n\Lambda + Br) + C\right]. \end{aligned}$$

Taking $\alpha$ large enough we can make $L\varphi \leqslant 0$ in $\Sigma$. Since $u(x) - u(x_0) < 0$ in $\Omega$, hence on $\partial B_\rho(y)$ which is compact, we have $\max_{x\in\partial B_\rho(y)}\left(u(x)-u(x_0)\right) < 0$ and there exists $\epsilon > 0$ such that $v(x) = u(x) - u(x_0) + \epsilon\varphi(x) \leqslant 0$ on $\partial B_\rho(y)$.

On the other hand, $\varphi = 0$ on $\partial B_r(y)$, therefore $v \leqslant 0$ on $\partial B_\rho(y) \cup \partial B_r(y) = \partial\Sigma$ and $Lv \leqslant -cu(x_0) \leqslant 0$.

So, by the weak maximum principle, $v \leqslant 0$ in $\bar{\Sigma}$. Being $v(x_0) = 0$ it follows that $\partial_\nu v(x_0) \geqslant 0$ and

$$\partial_\nu u(x_0) \geqslant -\epsilon\partial_\nu\varphi(x_0) = -\epsilon e^{\alpha|x_0-y|^2}|x_0-y| > 0.$$

□

PROOF OF THEOREM 5.3. Let $Lu \leqslant 0$ in $\Omega$ and assume by contradiction that $u$ is non-constant and achieves its maximum $M$ in the interior of $\Omega$ (with $M \geqslant 0$ if $c \neq 0$). Let $\Omega_M := \{y \in \Omega : u(y) = M\}$ and $\Omega^- := \{z \in \bar{\Omega} : u(z) < M\}$.

Let us consider $\bar{z} \in \Omega^-$ such that $r = \operatorname{dist}(\bar{z}, \Omega_M) < \operatorname{dist}(\bar{z}, \partial\Omega)$, so that $B_r(\bar{z}) \subset \Omega^-$ and there exists $x_0 \in \partial\Omega_M$ such that $x_0 \in \partial B_r(\bar{z})$.

Then we can apply Lemma 5.1 in $B_r(\bar{z})$. Indeed, all the hypotheses are verified: namely, $Lu \leqslant 0$ in $B_r(\bar{z}), u(x_0) > u(x)$, for all $x \in B_r(\bar{z})$ (with $u(x_0) = M \geqslant 0$ if $c \neq 0$), and $B_r(\bar{z})$ satisfies an interior sphere condition at $x_0$. We conclude therefore that $\partial_\nu u(x_0) > 0$ but this leads to a contradiction since $x_0$ is an interior maximum and $Du(x_0) = 0$. □

Beside needed to prove the strong maximum principle, Lemma 5.1 is also useful to establish uniqueness of solutions for Neumann problems.

COROLLARY 5.2 (Uniqueness for the Neumann problem). *Let $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be two solutions of the Neumann problem*

$$\begin{cases} Lu = f, & x \in \Omega, \\ \partial_\nu u = h, & x \in \partial\Omega, \end{cases}$$

*with $L$ uniformly elliptic in $\Omega$ and $\Omega$ satisfying an interior sphere condition at each point of $\partial\Omega$. If $c = 0$ in $\Omega$, then $u - v$ is constant in $\Omega$. If $c > 0$ at some point in $\Omega$, then $u = v$.*

PROOF. Suppose that $w = u - v$ is not constant in $\bar{\Omega}$. Since $Lw = Lu - Lv = 0$ in $\Omega$, then, by the strong maximum principle, either $w$ or $-w$ achieves a non-negative maximum $M$ at a point $x_0 \in \partial\Omega$ and is strictly less than $M$ in $\Omega$.

By Lemma 5.1, $\partial_\nu w(x_0) \neq 0$ (either strictly positive or strictly negative), contradicting the boundary condition $\partial_\nu w = 0$ on $\partial\Omega$. Hence, $w$ is constant in $\bar{\Omega}$ and, if $c \neq 0$ in $\Omega$, we have $w = 0$. □

**Lect. 14, 17.12**

## 2. Well-posedness of weak solutions of linear second-order elliptic PDEs

For $\mu \in \mathbb{R}$, we consider the Dirichlet problem for $L + \mu\mathrm{I}$,

$$\begin{cases} Lu + \mu u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

Here, I denotes the identity and we consider a uniformly elliptic operator $L$ in the (divergence) form

$$Lu := -\sum_{i,j=1}^{n} \partial_{x_i}\left(a_{ij}\partial_{x_j} u\right) + \sum_{i=1}^{n} \partial_{x_i}\left(b_i u\right) + cu, \tag{2.2}$$

where

$$a_{ij},\, b_i,\, c \in L^{\infty}(\Omega), \qquad a_{ij} = a_{ji} \tag{2.3}$$

and we assume that there exists some constant $\theta > 0$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \theta|\xi|^2, \tag{2.4}$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$.

We say that $u \in H_0^1(\Omega)$ is a *weak solution* of (2.1)

$$\int_\Omega \left\{ \sum_{i,j=1}^{n} a_{ij}\partial_{x_i}u\partial_{x_j}\phi - \sum_{i=1}^{n} b_i u\partial_{x_i}\phi + cu\phi \right\} \mathrm{d}x + \mu\int_\Omega u\phi\,\mathrm{d}x = \langle f, \phi\rangle$$

for all $\phi \in H_0^1(\Omega)$. To write this condition more concisely, we define a bilinear form

$$a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

by

$$a(u,v) := \int_\Omega \left\{ \sum_{i,j=1}^{n} a_{ij}\partial_{x_i}u\partial_{x_j}v - \sum_{i}^{n} b_i u\partial_{x_i}v + cuv \right\} \mathrm{d}x. \tag{2.5}$$

Definition 5.2 (Weak solution of (2.1)). *Let that $\Omega \subset \mathbb{R}^n$ be an open set, $f \in H^{-1}(\Omega)$, and $L$ be the differential operator (2.2). Then $u : \Omega \to \mathbb{R}$ is a weak solution of (2.1) if $u \in H_0^1(\Omega)$ and*

$$a(u,\phi) + \mu(u,\phi)_{L^2} = \langle f, \phi\rangle \qquad \text{for all } \phi \in H_0^1(\Omega).$$

Remark 5.2. *The form $a$ is well-defined and bounded on $H_0^1(\Omega)$. However, it is not symmetric unless $b_i = 0$. We have*

$$a(v,u) = a^*(u,v),$$

*where*

$$a^*(u,v) = \int_\Omega \left\{ \sum_{i,j=1}^{n} a_{ij}\partial_{x_i}u\partial_{x_j}v + \sum_{i}^{n} b_i\left(\partial_{x_i}u\right)v + cuv \right\} \mathrm{d}x \tag{2.6}$$

*is the bilinear form associated with the formal adjoint $L^*$ of $L$,*

$$L^*u = -\sum_{i,j=1}^{n} \partial_{x_i}\left(a_{ij}\partial_{x_j}u\right) - \sum_{i=1}^{n} b_i\partial_{x_i}u + cu. \tag{2.7}$$

Remark 5.3. *Notice that $L^*$ is the operator we considered in Section 1. We stress that, although $L^*$ is not of exactly the same form as $L$, since it first derivative term is not in divergence form, the same proof (up to minor changes) of the existence of weak solutions for $L$, which we will give below, applies to $L^*$ with $a$ in (2.5) replaced by $a^*$ in (2.6).*

*In the proof of the maximum principles in Section 1, we used $L^*$ for the sake of convenience (using $L$ instead would have required somewhat more attention).*

The proof of the existence of a weak solution of (2.1) is similar to the proof for the Dirichlet Laplacian, with one exception. If $L$ is not symmetric, we cannot use $a$ to define an equivalent inner product on $H_0^1(\Omega)$ and appeal to the Riesz representation theorem. Instead we use a result due to Peter Lax and Arthur Milgram (see [LM54] and also [Bab71]) which applies to non-symmetric bilinear forms.

We begin by stating the Lax–Milgram theorem for a bilinear form on a Hilbert space. Afterwards, we verify its hypotheses for the bilinear form associated with a general second-order uniformly elliptic PDE and use it to prove the existence of weak solutions.

THEOREM 5.4 (Lax–Milgram's theorem). *Let $\mathcal{H}$ be a Hilbert space with inner-product $(\cdot,\cdot) : \mathcal{H}\times\mathcal{H}\to\mathbb{R}$, and let $a : \mathcal{H}\times\mathcal{H}\to\mathbb{R}$ be a bilinear form on $\mathcal{H}$. Assume that there exist constants $C_1, C_2 > 0$ such that*

$$C_1\|u\|^2 \leqslant a(u,u), \quad |a(u,v)| \leqslant C_2\|u\|\|v\| \quad \text{for all } u,v\in\mathcal{H}.$$

*Then, for every bounded linear functional $f : \mathcal{H}\to\mathbb{R}$, there exists a unique $u\in\mathcal{H}$ such that*

$$\langle f,v\rangle = a(u,v) \quad \text{for all } v\in\mathcal{H}.$$

We omit the proof of this result, and instead we see how to apply it to the bilinear form in (2.5).

THEOREM 5.5. *Let $a$ be the bilinear form on $H_0^1(\Omega)$ defined in (2.5), where the coefficients satisfy (2.3) and the uniform ellipticity condition (2.4) with constant $\theta$. Then, there exist constants $C_1, C_2 > 0$ and $\gamma\in\mathbb{R}$ such that, for all $u$, $v\in H_0^1(\Omega)$,*

$$C_1\|u\|_{H_0^1}^2 \leqslant a(u,u) + \gamma\|u\|_{L^2}^2, \tag{2.8}$$

$$|a(u,v)| \leqslant C_2\|u\|_{H_0^1}\|v\|_{H_0^1}. \tag{2.9}$$

*If $b = 0$, we may take $\gamma := \theta - c_0$, with $c_0 := \inf_\Omega c$ and, if $b \neq 0$, we may take $\gamma := \frac{1}{2\theta}\sum_{i=1}^n \|b_i\|_{L^\infty}^2 + \frac{\theta}{2} - c_0$.*

REMARK 5.4. *Equation (2.8) is a crucial estimate of the $H_0^1$-norm of $u$ in terms of $a(u,u)$, using the uniform ellipticity of $L$, and is called Gårding's inequality.[1]. We notice that the expression for $\gamma$ given in Theorem 5.5 is not necessarily sharp. For example, as in the case of the Laplacian, the use of Poincare's inequality gives smaller values of $\gamma$ for bounded domains.*

*Equation (2.9) states that the bilinear form $a$ is bounded on $H_0^1$.*

PROOF OF THEOREM 5.5. Step 1. For any $u, v\in H_0^1(\Omega)$, we have

$$\begin{aligned}
|a(u,v)| &\leqslant \sum_{i,j=1}^n \int_\Omega |a_{ij}\partial_{x_i}u\partial_{x_j}v| \,\mathrm{d}x + \sum_{i=1}^n \int_\Omega |b_i u\partial_{x_i}v| \,\mathrm{d}x + \int_\Omega |cuv| \,\mathrm{d}x \\
&\leqslant \sum_{i,j=1}^n \|a_{ij}\|_{L^\infty} \|\partial_{x_i}u\|_{L^2} \left\|\partial_{x_j}v\right\|_{L^2} \\
&\quad + \sum_{i=1}^n \|b_i\|_{L^\infty} \|u\|_{L^2} \|\partial_{x_i}v\|_{L^2} + \|c\|_{L^\infty}\|u\|_{L^2}\|v\|_{L^2} \\
&\leqslant C\left(\sum_{i,j=1}^n \|a_{ij}\|_{L^\infty} + \sum_{i=1}^n \|b_i\|_{L^\infty} + \|c\|_{L^\infty}\right)\|u\|_{H_0^1}\|v\|_{H_0^1},
\end{aligned}$$

which yields (2.9).

Step 2. Using the uniform ellipticity condition (2.4), we have

$$\begin{aligned}
\theta\|Du\|_{L^2}^2 &= \theta\int_\Omega |Du|^2 \,\mathrm{d}x \\
&\leqslant \sum_{i,j=1}^n \int_\Omega a_{ij}\partial_{x_i}u\partial_{x_j}u \,\mathrm{d}x \\
&\leqslant a(u,u) + \sum_{i=1}^n \int_\Omega b_i u\partial_{x_i}u \,\mathrm{d}x - \int_\Omega cu^2 \,\mathrm{d}x \\
&\leqslant a(u,u) + \sum_{i=1}^n \int_\Omega |b_i u\partial_{x_i}u| \,\mathrm{d}x - c_0\int_\Omega u^2 \,\mathrm{d}x \\
&\leqslant a(u,u) + \sum_{i=1}^n \|b_i\|_{L^\infty} \|u\|_{L^2} \|\partial_{x_i}u\|_{L^2} - c_0\|u\|_{L^2} \\
&\leqslant a(u,u) + \beta\|u\|_{L^2}\|Du\|_{L^2} - c_0\|u\|_{L^2},
\end{aligned}$$

[1] Named after Lars Gårding [Går53]; cf. also [Hör18].

where $c(x) \geqslant c_0$ a.e. in $\Omega$, and $\beta = \left(\sum_{i=1}^n \|b_i\|_{L^\infty}^2\right)^{1/2}$.

If $\beta = 0$, we get (2.8) with

$$\gamma := \theta - c_0, \qquad C_1 := \theta.$$

On the other hand, if $\beta > 0$, by Youngs's inequality[2] with $\epsilon$, we have, for any $\epsilon > 0$,

$$\|u\|_{L^2}\|Du\|_{L^2} \leqslant \epsilon\|Du\|_{L^2}^2 + \frac{1}{4\epsilon}\|u\|_{L^2}^2.$$

Hence, choosing $\epsilon = \theta/2\beta$, we get

$$\frac{\theta}{2}\|Du\|_{L^2}^2 \leqslant a(u,u) + \left(\frac{\beta^2}{2\theta} - c_0\right)\|u\|_{L^2},$$

and (2.8) follows with

$$\gamma := \frac{\beta^2}{2\theta} + \frac{\theta}{2} - c_0, \qquad C_1 := \frac{\theta}{2}.$$

□

THEOREM 5.6 (Application to the Dirichlet problem). *Suppose that $\Omega$ is an open set in $\mathbb{R}^n$, and $f \in H^{-1}(\Omega)$. Let $L$ be a differential operator (2.2) (with coefficients satisfying (2.3)), and let $\gamma \in \mathbb{R}$ be a constant for which Theorem 5.5 holds. Then, for every $\mu \geqslant \gamma$, there exists one and only one weak solution of the Dirichlet problem (2.1).*

PROOF. For $\mu \in \mathbb{R}$, let us define $a_\mu : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by

$$a_\mu(u,v) = a(u,v) + \mu(u,v)_{L^2},$$

where $a$ is defined in (2.5). Then $u \in H_0^1(\Omega)$ is a weak solution of $Lu + \mu u = f$ if and only if

$$a_\mu(u,\phi) = \langle f, \phi\rangle \quad \text{for all } \phi \in H_0^1(\Omega).$$

We want to apply Lax–Milgram's theorem (Theorem 5.4) to establish the existence of one and only one such weak solution.

We rely on Theorem 5.5. From (2.9), we get

$$|a_\mu(u,v)| \leqslant C_2\|u\|_{H_0^1}\|v\|_{H_0^1} + |\mu|\|u\|_{L^2}\|v\|_{L^2} \leqslant (C_2 + |\mu|)\,\|u\|_{H_0^1}\|v\|_{H_0^1}$$

so $a_\mu$ is bounded on $H_0^1(\Omega)$. From (2.8),

$$C_1\|u\|_{H_0^1}^2 \leqslant a(u,u) + \gamma\|u\|_{L^2}^2 \leqslant a_\mu(u,u)$$

whenever $\mu \geqslant \gamma$.

Thus, the assumptions of Lax–Milgram's theorem are satisfied and we can conclude that, for every $f \in H^{-1}(\Omega)$, there is a unique $u \in H_0^1(\Omega)$ such that $\langle f,\phi\rangle = a_\mu(u,\phi)$ for all $v \in H_0^1(\Omega)$. This concludes the proof. □

[2] Named after William Henry Young [You12].

APPENDIX A

# Some background information

## 1. Some notation

Let $\mathbb{R}^n$ be $n$-dimensional Euclidean space. We denote the Euclidean norm of a vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ by

$$|x| = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2}$$

and the inner product of vectors $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n)$ by

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

We denote Lebesgue measure on $\mathbb{R}^n$ by $\mathrm{d}x$, and the Lebesgue measure of a set $E \subset \mathbb{R}^n$ by $|E|$.

If $E$ is a subset of $\mathbb{R}^n$, we denote the complement by $E^c = \mathbb{R}^n \backslash E$, the closure by $\bar{E}$, the interior by $E^\circ$ and the boundary by $\partial E = \bar{E} \backslash E^\circ$. The characteristic function $\chi_E : \mathbb{R}^n \to \mathbb{R}$ of $E$ is defined by

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

A set $E$ is bounded if $\{|x| : x \in E\}$ is bounded in $\mathbb{R}$. A set is *connected* if it is not the disjoint union of two non-empty relatively open subsets. We sometimes refer to a connected open set as a *domain*.

We say that an open set $\Omega' \subset \mathbb{R}^n$ is *compactly contained* in an open set $\Omega$, written $\Omega' \subset\subset \Omega$ (or $\Omega' \Subset \Omega$), if $\overline{\Omega'} \subset \Omega$ and $\overline{\Omega'}$ is compact. If $\bar{\Omega}' \subset \Omega$, then

$$\operatorname{dist}\left(\Omega', \partial\Omega\right) = \inf\left\{|x - y| : x \in \Omega', \, y \in \partial\Omega\right\} > 0.$$

This distance is finite provided that $\Omega' \neq \varnothing$ and $\Omega \neq \mathbb{R}^n$.

## 2. Integration by part formulas

We denote by $B_r(x) \subset \mathbb{R}^n$ the open ball of radius $r$ and center $x$ : $B_r(x) = \{y \in \mathbb{R}^n : \|y - x\| < r\}$ with respect to the Euclidean norm, and we will often use the shorthand notation $B_r = B_r(0)$. With $\mathbb{R}^n_+$ we denote the half-plane $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$.

Definition A.1. *Given a domain $\Omega \subset \mathbb{R}^n$, we say that $\partial\Omega$ is of class $C^k$ if, for each $y \in \partial\Omega$, there exists $r > 0$ and a $C^k$ function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, upon relabeling and reorienting the coordinates axis if necessary, we have*

$$\Omega \cap B_r(y) = \{x \in B_r(y) : x_n > \varphi(x_1, \cdots, x_{n-1})\}$$

*i.e. locally, in a neighborhood of $y \in \partial\Omega$, the boundary can be expressed as the graph of a $C^k$ function and the domain $\Omega$ lies only on one side of the graph. If $\partial\Omega \in C^1$, then $\forall y \in \partial\Omega$ the outward pointing normal vector is well defined and denoted $\nu(y) = (\nu_1(y), \cdots, \nu_n(y))$.*

We recall an equivalent characterization of a domain with $C^k$ boundary.

Proposition A.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $\partial\Omega$ of class $C^k$. Then for each $x \in \partial\Omega$ there exists an open set $V_x \ni x$ and a $C^k$-diffeomorphism $\phi_x : V_x \to B_1 \subset \mathbb{R}^n$ satisfying:*

*(1) $\phi_x(V_x \cap \Omega) = B_1^+ = B_1 \cap \mathbb{R}^n_+$,*
*(2) $\phi_x(V_x \cap \partial\Omega) = B_1 \cap \partial\mathbb{R}^n_+$,*
*(3) $\phi_x \in C^k(V_x)$, $\quad \phi_x^{-1} \in C^k(B_1)$.*

*We say that each mapping $\phi_x$ straightens locally the boundary.*

With the notion of the outward normal unit vector, we can now recall the Green–Gauss and integration by parts formulas.

**Gauss–Ostrogradsky's formula:** Let $\Omega \subset \mathbb{R}^n$ be a bounded set with $\partial\Omega \subset C^1$. For $u \in C^1(\bar{\Omega})$,

$$\int_\Omega u_{x_i}(x)\,\mathrm{d}x = \int_{\partial\Omega} u(y)\nu_i(y)\,\mathrm{d}S(y), \qquad i = 1, \cdots, n.$$

For more historical information of the *divergence theorem*, see [Kat79].

**Integration by parts formula:** Let $\Omega \subset \mathbb{R}^n$ be a bounded set with $\partial\Omega \subset C^1$. For $u, v \in C^1(\bar{\Omega})$,

$$\int_\Omega u_{x_i}(x)v(x)\,\mathrm{d}x = -\int_\Omega u(x)v_{x_i}(x)\,\mathrm{d}x + \int_{\partial\Omega} u(y)v(y)\nu_i(y)\,\mathrm{d}S(y), \quad i = 1, \cdots, n.$$

or, in vectorial form,

$$\int_\Omega \nabla u(x)v(x)\,\mathrm{d}x = -\int_\Omega u(x)\nabla v(x)\,\mathrm{d}x + \int_{\partial\Omega} u(y)v(y)\nu(y)\,\mathrm{d}S(y)$$

**Green's identities:** Let $\Omega \subset \mathbb{R}^n$ be a bounded set with $\partial\Omega \subset C^1$. For $u, v \in C^2(\bar{\Omega})$,

$$\int_\Omega \Delta u\,\mathrm{d}x = \int_{\partial\Omega} \partial_\nu u\,\mathrm{d}S,$$

with $\partial_\nu u = \nabla u \cdot \nu$ being the normal derivative of $u$, and

$$\int_\Omega v\Delta u\,\mathrm{d}x = -\int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x + \int_{\partial\Omega} v\partial_\nu u\,\mathrm{d}S = \int_\Omega u\Delta v\,\mathrm{d}x + \int_{\partial\Omega} (v\partial_\nu u - u\partial_\nu v)\,\mathrm{d}S.$$

All previous identities are valid also if the boundary $\partial\Omega$ is only Lipschitz continuous (since Lipschitz functions are differentiable everywhere but a set of points of Lebesgue measure zero).

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