

Transshipment

Problem definition and properties

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Introduction to optimization and operations research



The transshipment problem

Motivation

- ▶ The transshipment problem is probably the most typical network optimization problem.
- ▶ It is also known as the min-cost flow problem.
- ▶ We start by formulating it.

Motivation



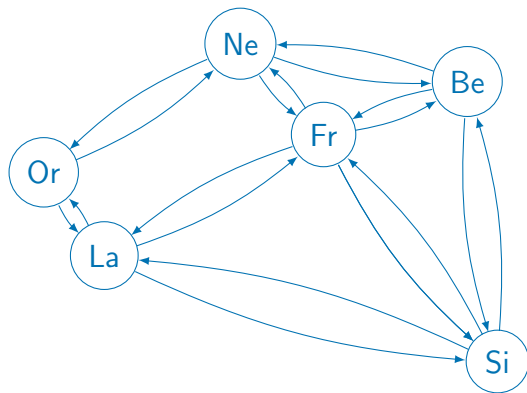
Logistics

- ▶ Goods have to be transported
 - ▶ from the warehouses,
 - ▶ to the clients.
- ▶ They can be transshipped anywhere on the network.
- ▶ Transportation costs must be minimum.

Data

Nodes

- Supply (warehouses).
- Demand (customers).
- Transshipment.



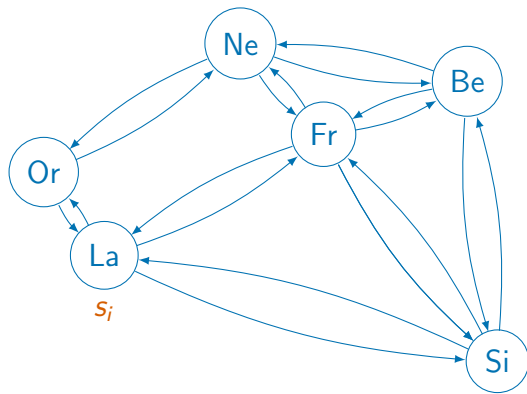
Data

Supply/demand

- ▶ $s_i > 0$ if supply node.
- ▶ $s_i < 0$ if demand node.
- ▶ $s_i = 0$ is transshipment node.

Validity

$$\sum_{i \in \mathcal{N}} s_i = 0.$$



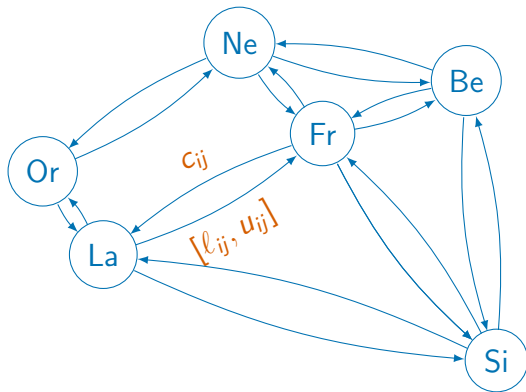
Data

Links

- Costs.
- Capacities.

Validity

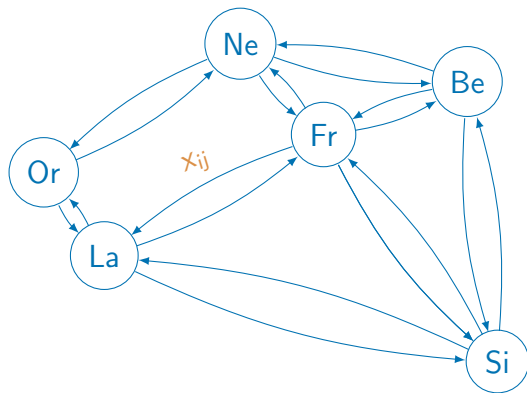
$$\ell_{ij} \leq u_{ij}.$$



Decision variables

Flow vector

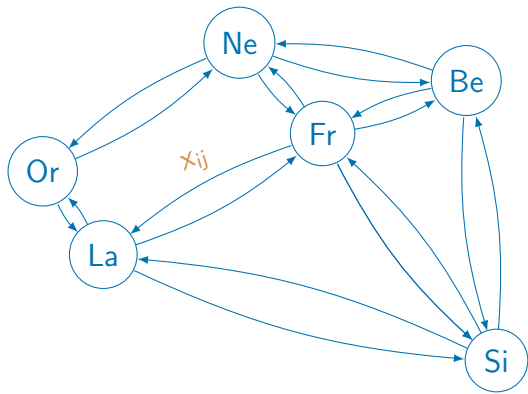
$$x \in \mathbb{R}^n.$$



Objective function

Total costs

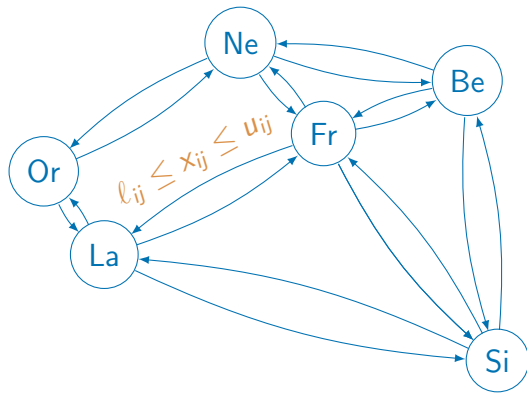
$$\sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}.$$



Constraints

Capacity

$$\ell_{ij} \leq x_{ij} \leq u_{ij}, \forall (i,j) \in \mathcal{A}.$$

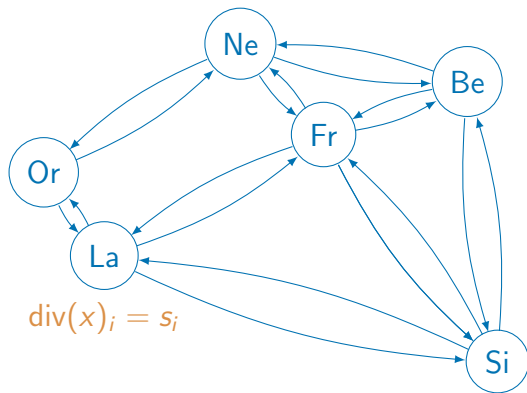


Constraints

Flow conservation

$$\text{div}(x)_i = s_i, \forall i \in \mathcal{N}.$$

$$\sum_{j|(i,j) \in \mathcal{A}} x_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x_{ki} = s_i, \forall i \in \mathcal{N}.$$



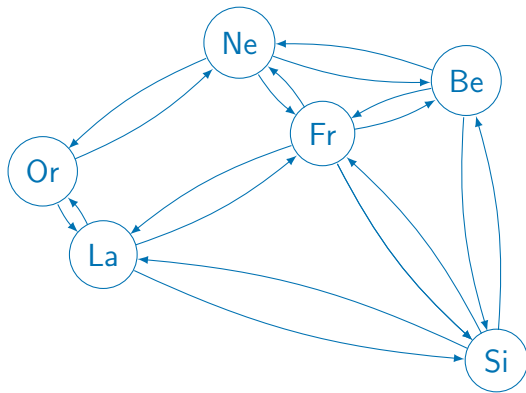
Linear optimization problem

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

subject to

$$\sum_{j|(i,j) \in \mathcal{A}} x_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x_{ki} = s_i, \forall i \in \mathcal{N}.$$

$$\ell_{ij} \leq x_{ij} \leq u_{ij}, \forall (i,j) \in \mathcal{A}.$$



Standard form

Motivation

- ▶ The transshipment problem is a linear optimization problem.
- ▶ The simplex algorithm that we have seen requires the problem to be written in standard form.
- ▶ We would like the problem in standard form to have an underlying network structure.

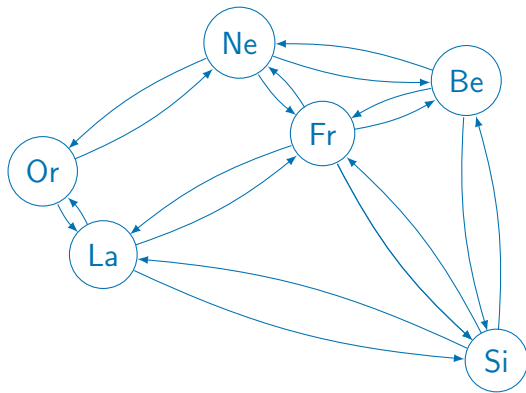
Linear optimization problem

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

subject to

$$\sum_{j|(i,j) \in \mathcal{A}} x_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x_{ki} = s_i, \forall i \in \mathcal{N}.$$

$$\ell_{ij} \leq x_{ij} \leq u_{ij}, \forall (i,j) \in \mathcal{A}.$$



Set the lower bounds to zero

$$x'_{ij} = x_{ij} - \ell_{ij} , \quad x_{ij} = x'_{ij} + \ell_{ij}$$

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x'_{ij} + \sum_{(i,j) \in \mathcal{A}} c_{ij} \ell_{ij} = \min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x'_{ij}$$

subject to

$$\sum_{j|(i,j) \in \mathcal{A}} x'_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x'_{ki} = s_i + \sum_{k|(k,i) \in \mathcal{A}} \ell_{ki} - \sum_{j|(i,j) \in \mathcal{A}} \ell_{ij} = s'_i, \quad \forall i \in \mathcal{C},$$

$$0 \leq x'_{ij} \leq u_{ij} - \ell_{ij} = u'_{ij}, \quad \forall (i,j) \in \mathcal{A}.$$

Slack variables

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x'_{ij}$$

subject to

$$\begin{aligned} \sum_{j|(i,j) \in \mathcal{A}} x'_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x'_{ki} &= s'_i \quad \forall i \in \mathcal{C}, \\ 0 \leq x'_{ij} &\leq u'_{ij}, \quad \forall (i,j) \in \mathcal{A}. \end{aligned}$$

Slack variables

$$0 \leq x'_{ij} \leq u'_{ij}, \quad \forall (i, j) \in \mathcal{A}.$$

Slack variables: y_{ij}

$$\begin{aligned} x'_{ij} + y_{ij} &= u'_{ij}, \\ x'_{ij} &\geq 0, \quad y_{ij} \geq 0. \end{aligned}$$

Interpretation

$$x'_{ij} + y_{ij} = u'_{ij}$$

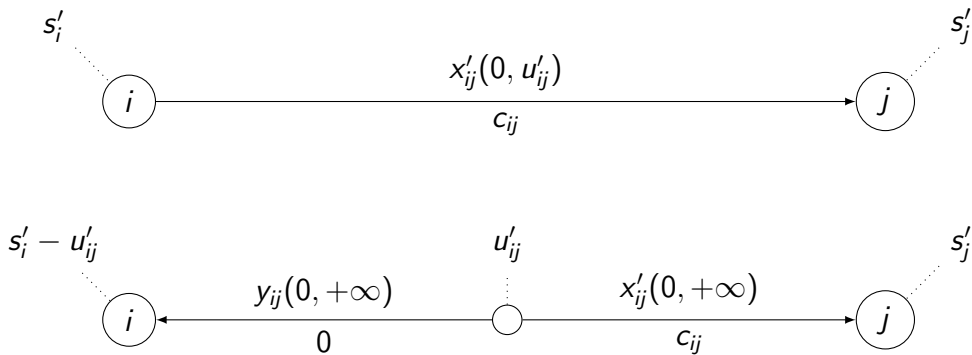
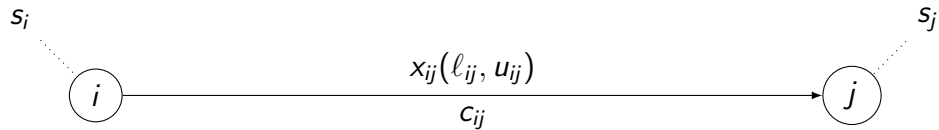
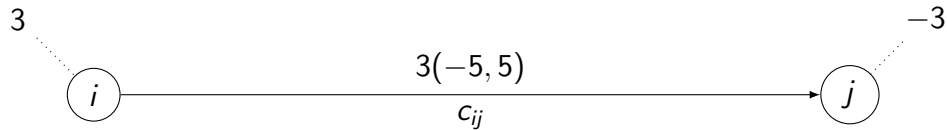


Illustration: initial formulation

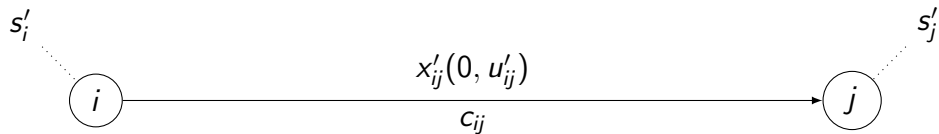


$$x_{ij} = s_i, \quad -x_{ij} = s_j, \quad \ell_{ij} \leq x_{ij} \leq u_{ij}.$$

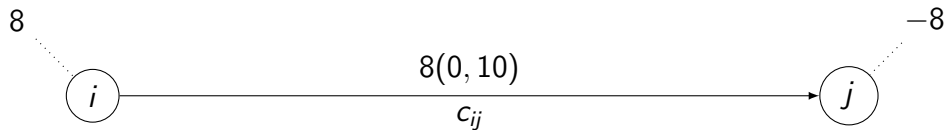


$$3 = 3, \quad -3 = -3, \quad -5 \leq 3 \leq 5.$$

Illustration: change of variables

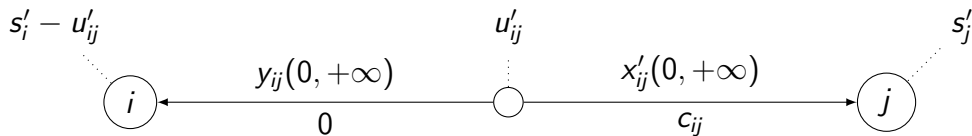


$$x_{ij} - \ell_{ij} = s_i - \ell_{ij}, \quad -x_{ij} + \ell_{ij} = s_j + \ell_{ij}, \quad 0 \leq x_{ij} - \ell_{ij} \leq u_{ij} - \ell_{ij}.$$

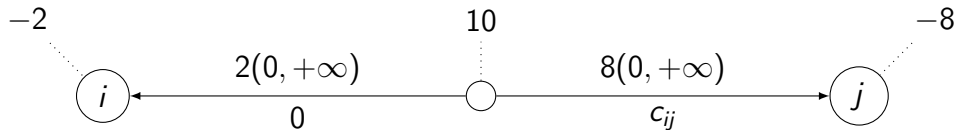


$$3 - (-5) = 3 - (-5), \quad -3 - 5 = -3 - 5, \quad 0 \leq 3 - (-5) \leq 5 - (-5).$$

Illustration: slack variables

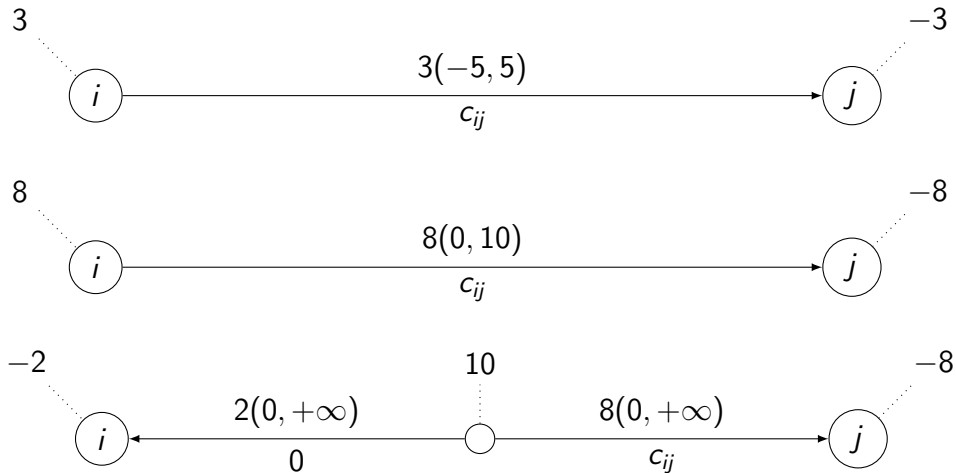


$$s'_i - u_{ij} = -y_{ij}, \quad x'_{ij} + y_{ij} = u'_{ij}, \quad s'_j = -x'_{ij}.$$



$$8 - 10 = -2, \quad 8 + 2 = 10, \quad -8 = -8.$$

Illustration: summary



Standard form

Comments

- ▶ Transforming the problem in standard form is equivalent to transforming the network.
- ▶ Main motivation: keep the structure of the matrix A .
- ▶ This can be done for any transshipment problem as pre-processing.
- ▶ Without loss of generality, we can therefore assume that the transshipment problem is written in standard form.

Problem in standard form: modified network

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

subject to

$$\sum_{j|(i,j) \in \mathcal{A}} x_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x_{ki} = s_i, \forall i \in \mathcal{N},$$
$$x_{ij} \geq 0, \forall (i,j) \in \mathcal{A}.$$

subject to

$$\min_{x \in \mathbb{R}^n} c^T x$$

$$Ax = s,$$

$$x \geq 0.$$

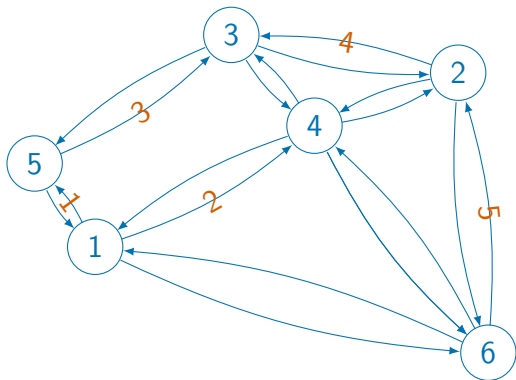
Incidence matrix

$$\sum_{j|(i,j) \in \mathcal{A}} x_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x_{ki} = s_i, \quad \forall i \in \mathcal{N}.$$

For each arc (i,j) of index k ,

$$a_{ik} = 1 \text{ and } a_{jk} = -1.$$

Incidence matrix



	1	2	3	4	5	...
1						
2						
3						
4						
5						
6						

Optimality conditions

Motivation

- ▶ The transshipment problem is a linear optimization problem, where the constraints have a special structure.
- ▶ We analyze now the dual problem and the dual variables.
- ▶ We introduce the optimality conditions called “complementarity slackness”.

Transshipment problem in standard form

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

subject to

$$\sum_{j|(i,j) \in \mathcal{A}} x_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x_{ki} = s_i, \forall i \in \mathcal{N},$$
$$x_{ij} \geq 0, \forall (i,j) \in \mathcal{A}.$$

subject to

$$\min_{x \in \mathbb{R}^n} c^T x$$

$$Ax = s,$$

$$x \geq 0.$$

Lagrangian

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

subject to

$$\sum_{j|(i,j) \in \mathcal{A}} x_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x_{ki} = s_i, \forall i \in \mathcal{N},$$
$$x_{ij} \geq 0, \forall (i,j) \in \mathcal{A}.$$

Lagrangian: $L(x, \lambda, \mu) =$

$$\sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$
$$+ \sum_{i \in \mathcal{N}} \lambda_i \left(\sum_{j|(i,j) \in \mathcal{A}} x_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x_{ki} - s_i \right)$$
$$- \sum_{(i,j) \in \mathcal{A}} \mu_{ij} x_{ij}.$$

Optimality conditions

Coefficients of x_{ij} must be 0:

$$\begin{aligned} L(x, \lambda, \mu) = & \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij} \\ & + \sum_{i \in \mathcal{N}} \lambda_i \left(\sum_{j | (i,j) \in \mathcal{A}} x_{ij} - \sum_{k | (k,i) \in \mathcal{A}} x_{ki} - s_i \right) \\ & - \sum_{(i,j) \in \mathcal{A}} \mu_{ij} x_{ij}, \end{aligned}$$
$$\begin{aligned} \frac{\partial L}{\partial x_{ij}} = & c_{ij} + \lambda_i - \lambda_j - \mu_{ij} = 0. \\ & \mu_{ij} = c_{ij} + \lambda_i - \lambda_j \\ & \mu_{ij} \geq 0 \text{ and } \mu_{ij} x_{ij} = 0 \end{aligned}$$

Complementarity slackness

For all (i, j)

$$c_{ij} + \lambda_i - \lambda_j \geq 0.$$

For all (i, j) such that $x_{ij} > 0$

$$c_{ij} + \lambda_i - \lambda_j = 0.$$

Sufficient and necessary optimality conditions.

Total unimodularity

Motivation

- ▶ The incidence matrix defining the constraints of a transshipment problem has a special structure.
- ▶ It also has a great property called “total unimodularity”.
- ▶ It is great because it guarantees to generate an integer solution, if the data is integer.
- ▶ We now formally define this property.

Transshipment problem in standard form

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

subject to

$$\sum_{j|(i,j) \in \mathcal{A}} x_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x_{ki} = s_i, \forall i \in \mathcal{N},$$
$$x_{ij} \geq 0, \forall (i,j) \in \mathcal{A}.$$

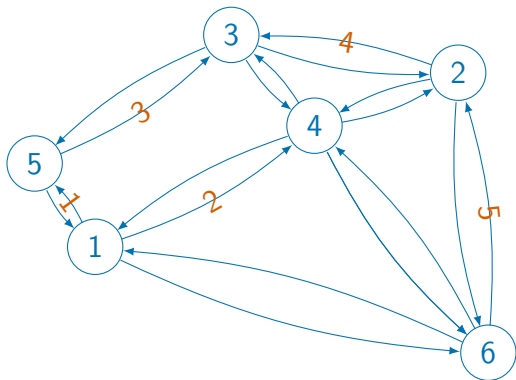
subject to

$$\min_{x \in \mathbb{R}^n} c^T x$$

$$Ax = s,$$

$$x \geq 0.$$

Incidence matrix



	1	2	3	4	5	...
1	1	1	0	0	0	
2	0	0	0	1	-1	
3	0	0	-1	-1	0	
4	0	-1	0	0	0	
5	-1	0	1	0	0	
6	0	0	0	0	1	

Optimal solution

$$x_B^* = B^{-1}s.$$

Cramer's rule

$$x_B^* = \frac{1}{\det(B)} C(B)^T s$$

Each entry of the matrix is 0, 1 or -1.

Each entry of $C(B)$ is an integer.

If s is integer, $C(B)^T s$ is integer.

If $\det(B)$ is 1 or -1, x_b^* is integer.

Total unimodularity

Definition

The matrix $A \in \mathbb{Z}^{m \times n}$ is **totally unimodular** if the determinant of each square submatrix of A is

$$0, -1 \text{ or } +1.$$

Total unimodularity

Definition

The matrix $A \in \mathbb{Z}^{m \times n}$ is **totally unimodular** if the determinant of each square submatrix of A is

$$0, -1 \text{ or } +1.$$

In particular,

every entry of the matrix is 0, -1 or $+1$.

Example

Each square submatrix of size 1, that is each element of the matrix, is 0, -1 or $+1$. Each square submatrix of size 2 is unimodular:

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

$$\det \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = 1,$$

$$\det \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = -1,$$

$$\det \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = 1.$$

Integrality of the basic solutions

Consider the polyhedron represented in standard form

$$\{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\},$$

where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$.

If A is totally unimodular, then every basic solution is integer.

Proof

$$x_B = B^{-1}b. \quad x_B = \frac{1}{\det(B)} C(B)^T b,$$

Total unimodularity of the incidence matrix

Motivation

- ▶ Total unimodularity of the matrix defining the constraints of a linear optimization problem guarantees that the basic solutions are integer if the data are integer.
- ▶ Therefore, the optimal solution is also integer.
- ▶ This is a great property, given the difficulty of discrete optimization.
- ▶ We show now that the incidence matrix of the transshipment problem is totally unimodular.

Transshipment problem in standard form

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

subject to

$$\sum_{j|(i,j) \in \mathcal{A}} x_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x_{ki} = s_i, \forall i \in \mathcal{N},$$
$$x_{ij} \geq 0, \forall (i,j) \in \mathcal{A}.$$

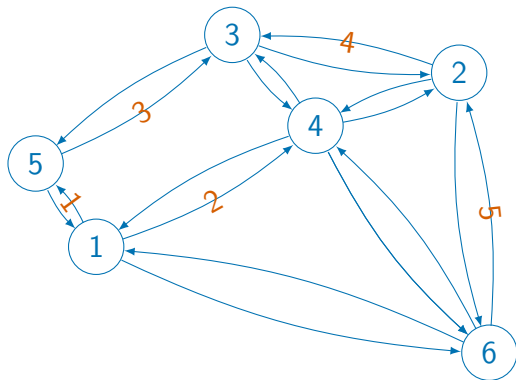
subject to

$$\min_{x \in \mathbb{R}^n} c^T x$$

$$Ax = s,$$

$$x \geq 0.$$

Incidence matrix



	1	2	3	4	5	...
1	1	1	0	0	0	
2	0	0	0	1	-1	
3	0	0	-1	-1	0	
4	0	-1	0	0	0	
5	-1	0	1	0	0	
6	0	0	0	0	1	

Theorem 22.6

The incidence matrix of a network is totally unimodular.

Proof

- ▶ By contradiction, assume that it is not the case.
- ▶ There exist square matrices with determinant $\notin \{0, 1, -1\}$.
- ▶ Among all of them, consider the one with minimum size k : B .
- ▶ Note that $k > 1$.
- ▶ Number of non-zero entries in a column of B : 0, 1 or 2.
- ▶ We treat each case separately.

No non-zero entry

Context

- ▶ B is $k \times k$.
- ▶ $\det(B) \notin \{0, 1, -1\}$.
- ▶ $\forall \ell \times \ell$ submatrices B' : $\det(B') \in \{0, 1, -1\}$.
- ▶ One column of B contains only zeros.

Impossible, because it would mean that $\det(B) = 0$.
Therefore, no column of B contains only 0s.

One non-zero entry

Context

- ▶ B is $k \times k$.
- ▶ $\det(B) \notin \{0, 1, -1\}$.
- ▶ $\forall \ell \times \ell$ submatrices B' : $\det(B') \in \{0, 1, -1\}$.
- ▶ One column of B contains only one non-zero entry.

$$B = \begin{pmatrix} b_{11} & b_{12} \cdots b_{1k} \\ 0 & \\ \vdots & B' \\ 0 & \end{pmatrix}$$

$$\det(B) = b_{11} \det(B') = \det(B') \text{ or } -\det(B')$$

Impossible as B is the smallest matrix.

Two non-zero entries

Context

- ▶ B is $k \times k$.
- ▶ $\det(B) \notin \{0, 1, -1\}$.
- ▶ $\forall \ell \times \ell$ submatrices B' : $\det(B') \in \{0, 1, -1\}$.
- ▶ Each column of B contains exactly two non-zero entries.

The two non zero entries are 1 and -1.

If you sum the elements of each column, it is equal to zero.

So, summing all the rows of B gives 0

B is singular, and its determinant is zero.

Contradiction.

Corollary

If the supply vector s of the transshipment problem is integer, that is, if $s \in \mathbb{Z}^m$, and if the problem is bounded, then it has an optimal solution which is integer.

The shortest path problem

Motivation

- ▶ The transshipment problems embeds a variety of other problems.
- ▶ These problems inherit the properties of the transshipment problem.
- ▶ In particular, the integrality of the optimal solution if the data is integer.
- ▶ We first introduce the shortest path problem as an instance of the transshipment problem.

Shortest path



Problem definition

- ▶ Consider a network with nodes and arcs.
- ▶ Each arc is associated with a cost.
- ▶ Consider an origin and a destination.
- ▶ Find a simple forward path from o to d with the smallest cost.

Shortest paths problems

- ▶ Single origin – single destination. (this lecture)
- ▶ Single origin – all destinations. (next topic)
- ▶ All origins – single destination.
- ▶ All origins – all destinations.

Cost of a path

Definition

$$C(P) = \sum_{(i,j) \in P^{\rightarrow}} c_{ij} x_{ij} - \sum_{(i,j) \in P^{\leftarrow}} c_{ij} x_{ij}.$$

Forward path

$$C(P) = \sum_{(i,j) \in P^{\rightarrow}} c_{ij} x_{ij} = \sum_{(i,j) \in P} c_{ij} x_{ij}$$

Transshipment

Flow

Send one unit of flow from o to d .

Supply at o : $s_o = 1$

Supply at d : $s_d = -1$

Supply at any other node: $s_i = 0$

Total unimodularity: if there is a solution, flow on (i, j) : 0 or 1

Cost of a path

$$C(P) = \sum_{(i,j) \in P} c_{ij} x_{ij} = \sum_{(i,j) \in P} c_{ij}$$

Transshipment

- ▶ cost on each arc: c_{ij}
- ▶ supply at the origin: $s_o = 1$
- ▶ supply at the destination: $s_d = -1$
- ▶ supply at any other node: 0
- ▶ lower bound on each arc: 0
- ▶ upper bound on each arc: ∞

Transshipment

$$\min_{x \in \mathbb{R}^n} \sum_{(i,j) \in \mathcal{A}} c_{ij} x_{ij}$$

subject to

$$\sum_{j|(o,j) \in \mathcal{A}} x_{oj} - \sum_{k|(k,o) \in \mathcal{A}} x_{ko} = 1,$$

$$\sum_{j|(d,j) \in \mathcal{A}} x_{dj} - \sum_{k|(k,d) \in \mathcal{A}} x_{kd} = -1,$$

$$\sum_{j|(i,j) \in \mathcal{A}} x_{ij} - \sum_{k|(k,i) \in \mathcal{A}} x_{ki} = 0, \quad \forall i \in \mathcal{N}, i \neq o, i \neq d,$$

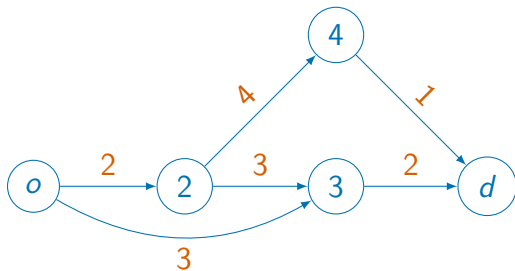
$$x_{ij} \geq 0, \quad \forall (i,j) \in \mathcal{A}.$$

The maximum flow problem

Motivation

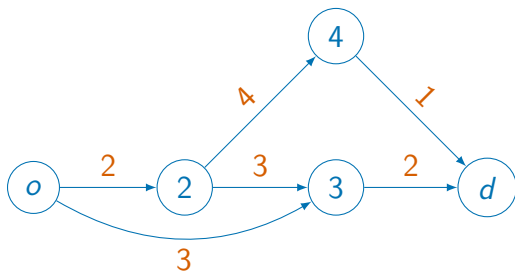
- ▶ The maximum flow problem was motivated by the analysis performed by the American army of the railway network operated by the Soviet Union across Eastern Europe during the Cold War.
- ▶ We have a network, where each arc has a capacity.
- ▶ Consider an origin o and a destination d .
- ▶ What is the maximum amount of flow that the network can carry from o to d ?

The maximum flow problem

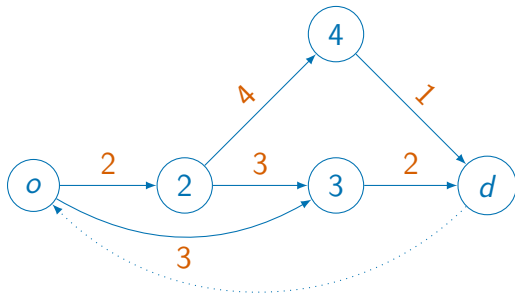


- ▶ $o \rightarrow 2 \rightarrow 4 \rightarrow d$: 1 unit of flow.
- ▶ $o \rightarrow 3 \rightarrow d$: 2 units of flow.
- ▶ $o \rightarrow 2 \rightarrow 3 \rightarrow d$: saturated.

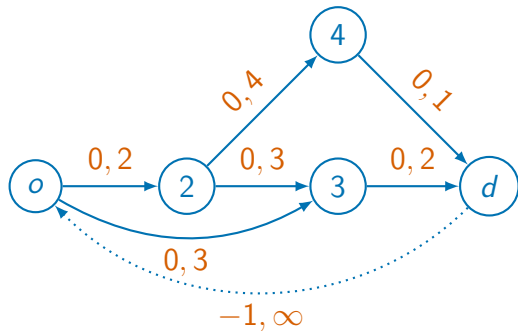
Transshipment



Transshipment



Transshipment



- ▶ cost on the artificial arc: -1
- ▶ cost on every other arc: 0
- ▶ supply/demand for each node: 0
- ▶ lower bound on each arc: 0
- ▶ upper bound on the artificial arc: ∞
- ▶ upper bound on every other arc: from the problem definition

Transshipment

$$\min_{x \in \mathbb{R}^{n+1}} -x_{do}$$

subject to

$$\sum_{j|(i,j) \in \mathcal{AU}(d,o)} x_{ij} - \sum_{k|(k,i) \in \mathcal{AU}(d,o)} x_{ki} = 0, \quad \forall i \in \mathcal{N}, \quad (1)$$

$$x_{ij} \leq u_{ij}, \quad \forall (i,j) \in \mathcal{A}, \quad (2)$$

$$x_{ij} \geq 0, \quad \forall (i,j) \in \mathcal{A}, \quad (3)$$

$$x_{do} \geq 0. \quad (4)$$

The transportation problem

Motivation

- ▶ The transportation problem is somehow a transshipment problem where transshipment is not allowed.
- ▶ We have suppliers and customers.
- ▶ There is a cost to transport items from a supplier to a customer.
- ▶ We need to decide how much each supplier will deliver to each customer.

Electricity in Switzerland



Suppliers

- ▶ Mühleberg: 3110 GWh/year
- ▶ Beznau: 3198 GWh/year
- ▶ Leibstadt: 10205 GWh/year

Customers

- ▶ Zürich: 8961 GWh/year
- ▶ Geneva: 3777 GWh/year
- ▶ Lausanne: 2517 GWh/year
- ▶ Bern: 1258 GWh/year

Electricity in Switzerland

Unit costs

	Zürich	Geneva	Lausanne	Bern
Mühleberg	18	6	10	9
Beznau	9	16	13	7
Leibstadt	14	9	16	5



Network

Comments

- ▶ No “real” network.
- ▶ We create a network representation.
- ▶ One set of nodes for the suppliers.
- ▶ One set of nodes for the customers.

Network

Leibstadt ●

Beznau ●

Mühleberg ●

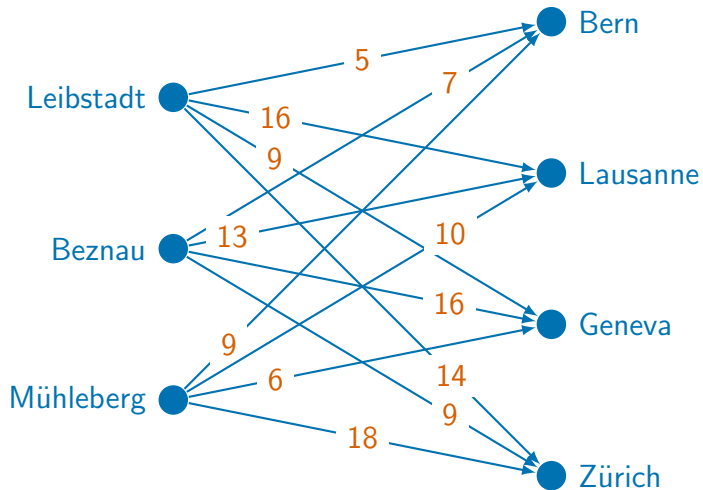
● Bern

● Lausanne

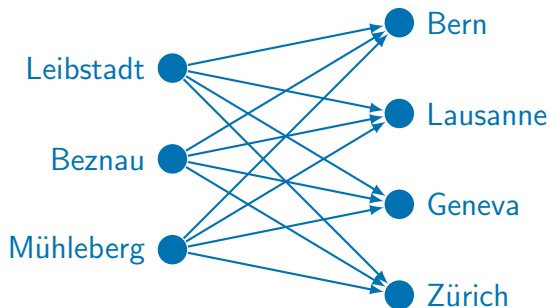
● Geneva

● Zürich

Network



Transshipment



- ▶ cost on each arc: c_{ij}
- ▶ supply for supplier nodes: s_i
- ▶ supply for customer nodes: $-t_j$
- ▶ lower bound on each arc: 0
- ▶ upper bound on each arc: $+\infty$

Transshipment

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{m_o} \sum_{j=1}^{m_d} c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{m_d} x_{ij} = s_i, \quad i = 1, \dots, m_o,$$

$$\sum_{i=1}^{m_o} x_{ij} = t_j, \quad j = 1, \dots, m_d,$$

$$\begin{aligned} x_{ij} &\geq 0, & i = 1, \dots, m_o, \quad j = 1, \dots, m_d, \\ x_{ij} &= 0, & \text{if } i \text{ does not serve } j. \end{aligned}$$

The assignment problem

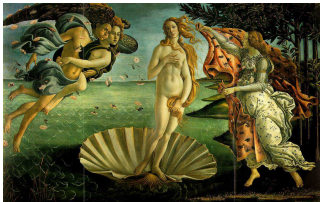
Motivation

The assignment problem consists in assignment n tasks to n resources at minimal cost.

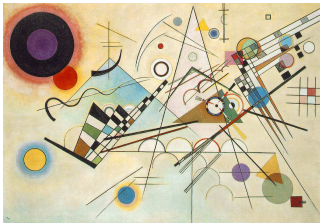
Heirs

- ▶ After the death of her husband, my grandmother has discovered four masterpieces in her attic.
- ▶ She does not want to keep them.
- ▶ She would like to sell each of them to one of her four children.
- ▶ Each child made an offer for the masterpieces of interest.
- ▶ She wants to sell exactly one masterpiece to each child.
- ▶ Which masterpiece should she sell to which child, to maximize her revenues?

Heirs



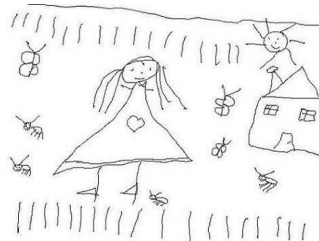
Botticelli, 1485



Kandinsky, 1923



Bruegel, 1558



Bierlaire, 1971

Offers (kEuros)

	Botticelli	Bruegel	Kandinsky	Bierlaire
Harry	8000	11000	—	—
Ron	9000	13000	12000	—
Hermione	9000	—	11000	0.01
Ginny	—	14000	12000	—

Network



● Harry



● Ron

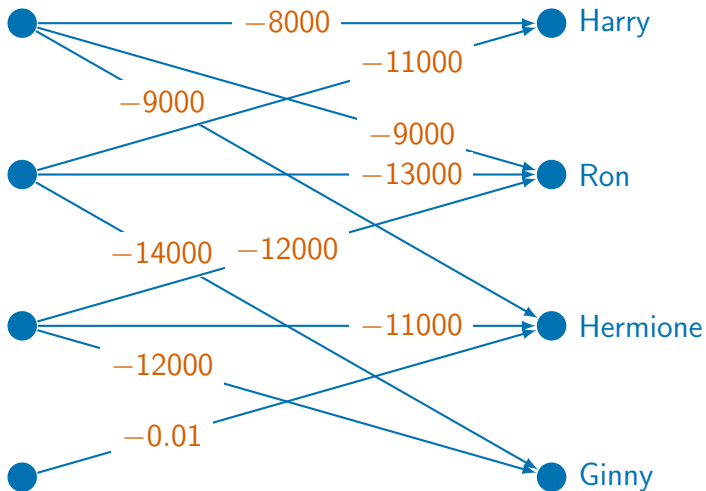


● Hermione

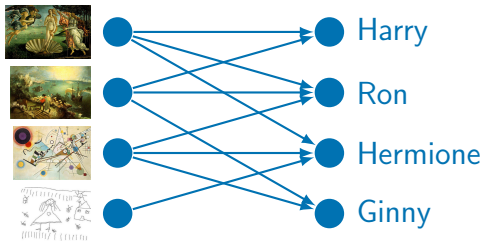


● Ginny

Network



Transshipment



- ▶ cost on each arc: $c_{ij} = -p_{ij}$
- ▶ supply for each resource node: 1
- ▶ supply for each task node: -1
- ▶ lower bound on each arc: 0
- ▶ upper bound on each arc: 1

Transshipment

$$\min_{x \in \mathbb{R}^{n^2}} \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^n x_{ij} = 1, \quad i = 1, \dots, n, \quad (5)$$

$$\sum_{i=1}^n x_{ij} = 1, \quad j = 1, \dots, n, \quad (6)$$

$$x_{ij} \geq 0, \quad i = 1, \dots, n, \quad j = 1, \dots, n, \quad (7)$$

$$x_{ij} \leq 1, \quad i = 1, \dots, n, \quad j = 1, \dots, n. \quad (8)$$

The variable x_{ij} is to take on the value 1 if resource i is assigned to task j , and 0 otherwise.

Summary

- ▶ Transshipment problem: linear optimization problem.
- ▶ Special structure.
- ▶ Property: total unimodularity of the incidence matrix.
- ▶ Consequence: if the data is integer, the optimal solution is also integer.
- ▶ Shortest path problem.
- ▶ Maximum flow problem.
- ▶ Transportation problem.
- ▶ Assignment problem.