Networks

Introduction and definitions

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Introduction to optimization and operations research

Definitions

Motivation

- Networks are everywhere.
- We introduce a mathematical formalism that mimics the structure of real networks.

Road networks

Public transportation networks

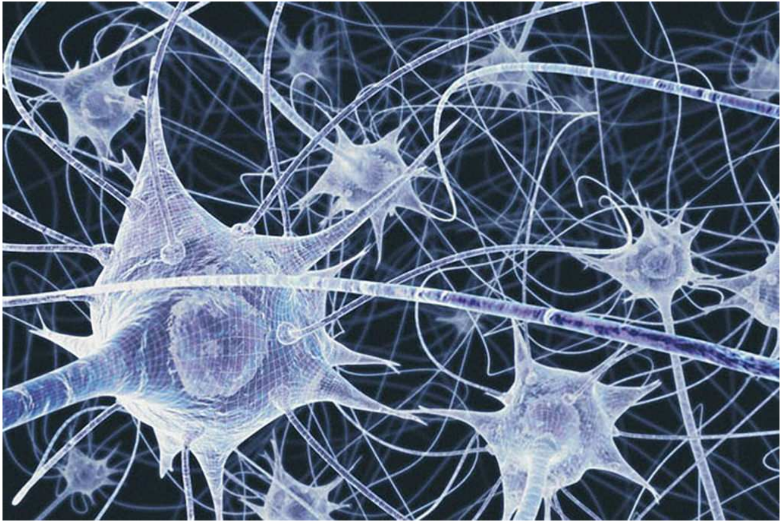
Electricity networks

Gas networks

Water networks

Computer networks

Neural networks



Social networks

Networks

Concept

System of interconnected people or things.

Main features

- Local complexity is low.
- Global complexity is high.

Mathematical object

Similar property: designed to capture complex structures with simple elements.

Networks

Concept

System of interconnected people or things.

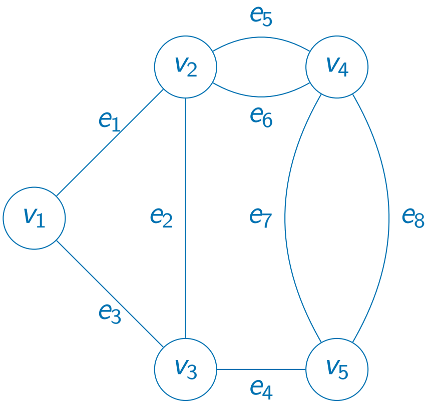
Vocabulary

- people or thing: vertex, node.
- connection: edge: link (undirected), arc (directed).
- structure: graph (no data), network (with data).

Undirected graph

Definition

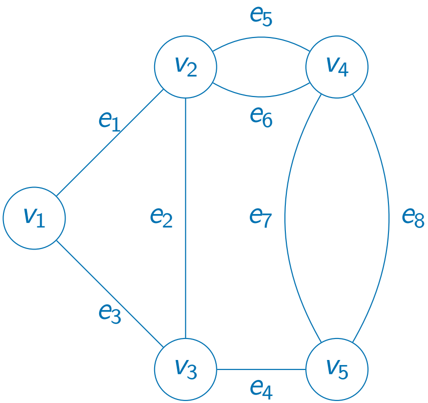
- $\mathcal{V}$: set of vertices.
- $\mathcal{E}$: set of edges.
- $\phi : \mathcal{E} \to \mathcal{P}_2(\mathcal{V})$: incidence function.



Graph

Example

$$
\begin{aligned}
\mathcal{V} &= \{v_1, v_2, v_3, v_4, v_5\}, \\
\mathcal{E} &= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\} \\
\phi(e_1) &= \{v_1, v_2\},\ \phi(e_2) = \{v_2, v_3\}, \\
\phi(e_3) &= \{v_1, v_3\},\ \phi(e_4) = \{v_3, v_5\}, \\
\phi(e_5) &= \{v_2, v_4\},\ \phi(e_6) = \{v_2, v_4\}, \\
\phi(e_7) &= \{v_4, v_5\},\ \phi(e_8) = \{v_4, v_5\}.
\end{aligned}
$$

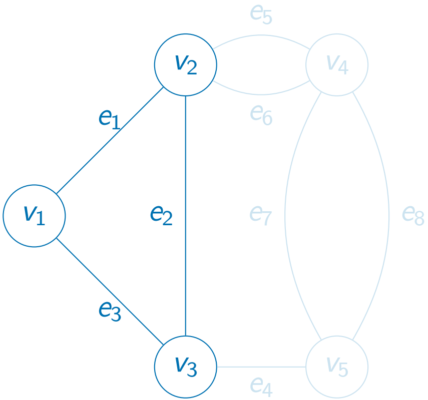


Subgraph

Definition

$(\mathcal{V}', \mathcal{E}', \phi')$ is a subgraph of $(\mathcal{V}, \mathcal{E}, \phi)$ if

- $\mathcal{V}' \subseteq \mathcal{V}$,
- $\mathcal{E}' \subseteq \mathcal{E}$,
- $\phi'(e) = \phi(e)$, for each $e \in \mathcal{E}'$,
- for each $e \in \mathcal{E}'$, if $\phi'(e) = \{i, j\}$, then i and j both belong to $\mathcal{V}'$.



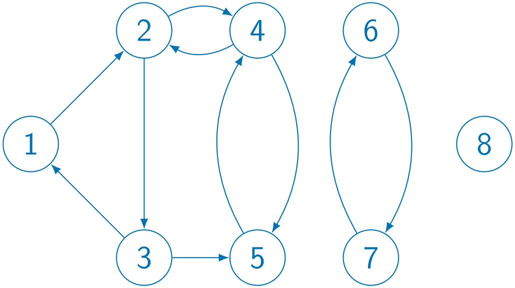
Directed graph

Definition

- $\mathcal{N}$: set of nodes.
- $\mathcal{A}$: set of arcs.
- $\phi : \mathcal{A} \to \mathcal{N} \times \mathcal{N}$: incidence function.

Assumption

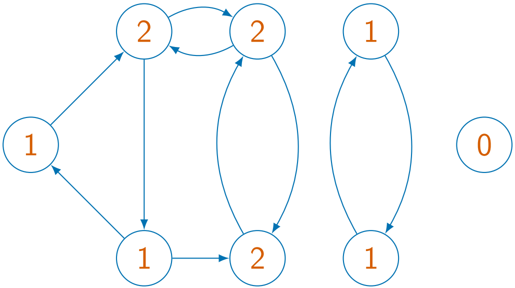
- ϕ is injective.
- For each (i, j), there is at most one a such that $\phi(a) = (i, j)$.
- Arcs are denoted by (i, j).



Indegree

Definition

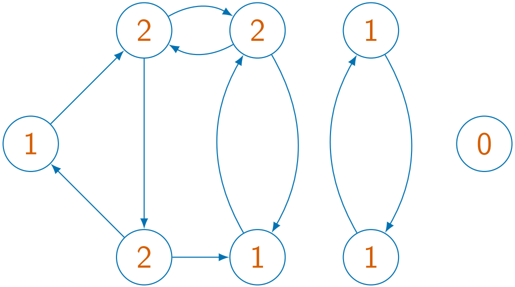
d_i^-: number of arcs (j, i).



Outdegree

Definition

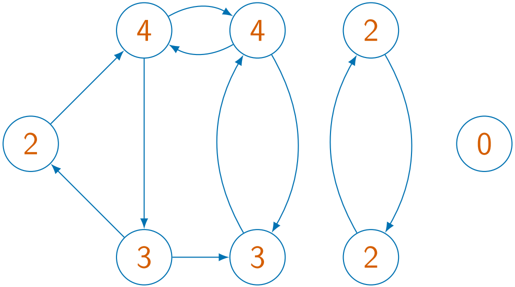
d_i^+: number of arcs (i, j).



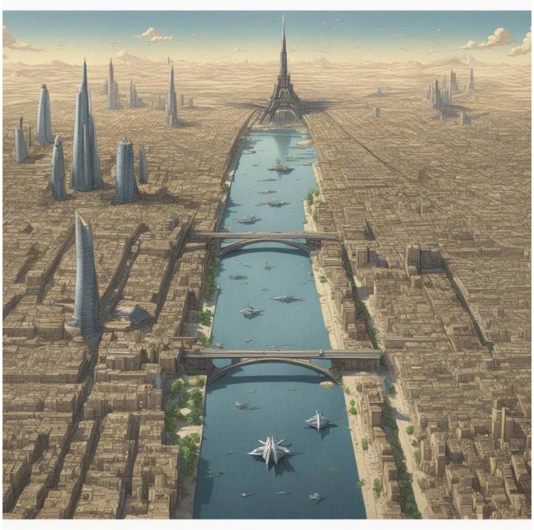
Degree

Definition

$$d_i = d_i^- + d_i^+.$$



Cuts



Motivation

- Just as cities are separated into two banks by a river, it may be convenient to separate a directed graph into two sets of nodes.
- This is called a cut.

Directed graph: $(\mathcal{N}, \mathcal{A}, \phi)$

Cut

A *cut* Γ is

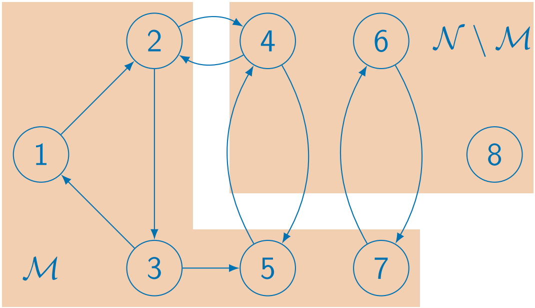
- an ordered partition of the nodes
- into two non empty subsets:

$$\Gamma = (\mathcal{M}, \mathcal{N} \setminus \mathcal{M}),$$

where $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{M} \neq \emptyset$.

Ordered

$$(\mathcal{M}, \mathcal{N} \setminus \mathcal{M}) \neq (\mathcal{N} \setminus \mathcal{M}, \mathcal{M})$$



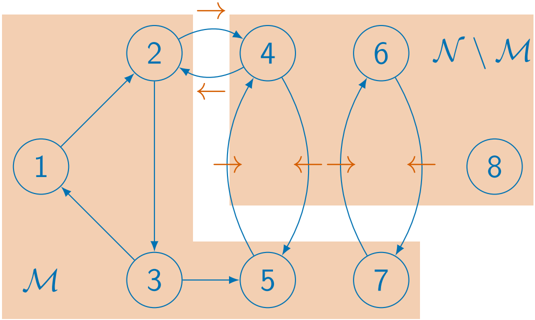
Definitions

Forward arcs

$$\Gamma^{\rightarrow} = \{(i,j) \in \mathcal{A} | i \in \mathcal{M}, j \notin \mathcal{M}\}.$$

Backward arcs

$$\Gamma^{\leftarrow} = \{(i,j) \in \mathcal{A} | i \notin \mathcal{M}, j \in \mathcal{M}\}.$$



Paths

Motivation

- Networks are designed to connect elements.
- The concept of paths describe how two elements in the network can be connected to each other.

Definition

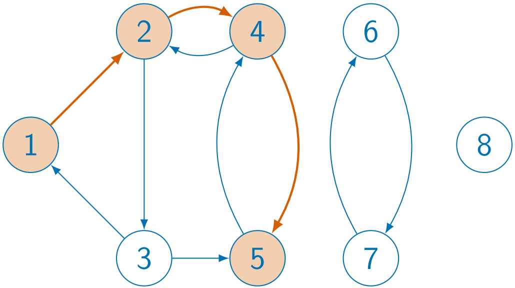
- Sequence of nodes, each pair of consecutive nodes being directed with forward or backward.
- Simple path: no repeated node.
- Forward path: no backward arc.

Simple forward path

$$1 \to 2 \to 4 \to 5$$

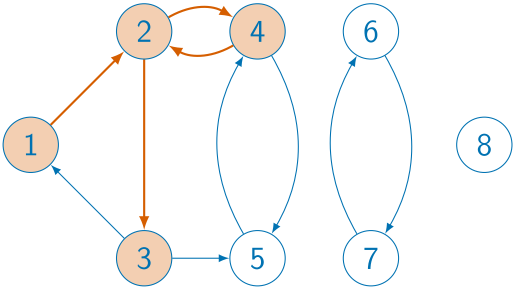
$$P^{\to} = (1,2),(2,4),(4,5),$$

$$P^{\leftarrow} = \emptyset$$



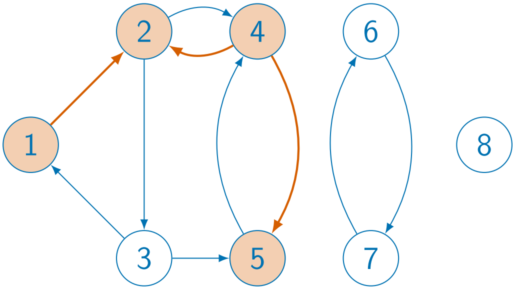
Forward path

$$1 \to 2 \to 4 \to 2 \to 3$$
$$P^{\to} = (1,2),(2,4),(4,2),(2,3),$$
$$P^{\leftarrow} = \emptyset$$



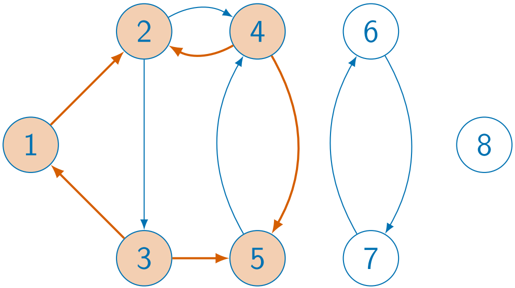
Simple path

$$1 \rightarrow 2 \leftarrow 4 \rightarrow 5$$
$$P^{\rightarrow} = (1, 2), (4, 5)$$
$$P^{\leftarrow} = (4, 2)$$



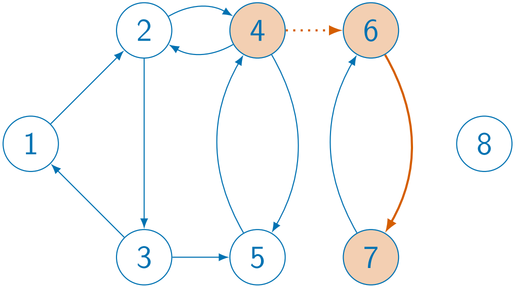
Simple cycle

$$1 \to 2 \leftarrow 4 \to 5 \leftarrow 3 \to 1$$
$$P^{\to} = (1, 2), (4, 5), (3, 1)$$
$$P^{\leftarrow} = (4, 2), (3, 5)$$



Invalid path

$4 \to 6 \to 7$



Paths and connected components

Motivation

- There are many paths in a network.
- Some are long, some are short.
- Later on, we will be interested in finding the shortest or the longest path between two nodes.
- And, sometimes, there is no path connecting two nodes.
- We formalize these concepts now.

Longest simple path

Lemma 21.5

- Consider a directed graph with m nodes.
- The maximum number of arcs in a simple path is $m - 1$.

Proof

- Suppose that a simple path visiting all the m nodes exist.
- It has exactly $m - 1$ arcs.
- Extend by one more arc: not simple anymore.
- If there is no such path: the longest has less than $m - 1$ arcs.

Finite number of simple paths

Lemma 21.6

- Consider a directed graph with m nodes, $m \geq 2$.
- Consider an origin node o and a destination node d.
- There is a finite number of simple paths between o and d.

Proof

- Consider $2 \leq k \leq m$.
- Each simple path containing k nodes corresponds to a permutation of $k - 2$ nodes.
- For each k, the number of permutations is finite.
- As $k \leq m$, the total number is finite.

Connectivity

Connected graph

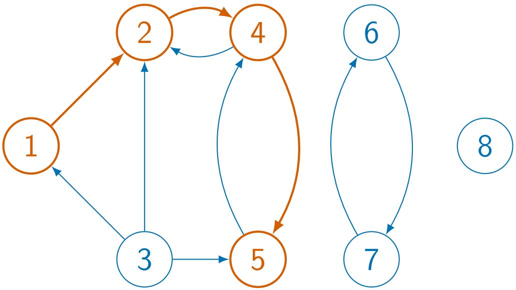
Every pair of nodes is connected with a path.

Strongly connected graph

Every pair of nodes is connected with a path containing only forward arcs.

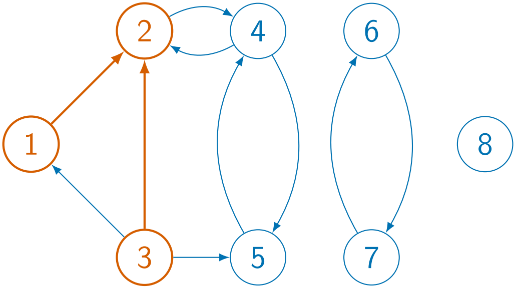
Connectivity

Nodes 1 and 5 are strongly connected.



Connectivity

Nodes 1 and 3 are connected.



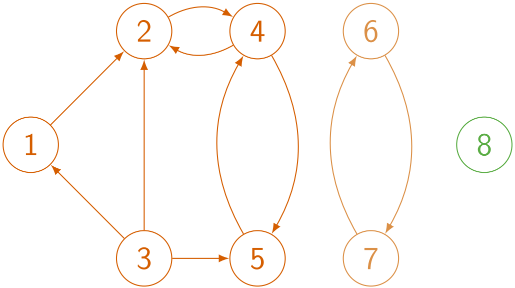
Equivalence class

“is connected with”

- reflexive,
- symmetric,
- transitive.

Connected component

- subgraph $G' = (\mathcal{N}', \mathcal{A}', \phi')$,
- $\mathcal{N}'$ is an equivalence class on $\mathcal{N}$ for the relation “is connected with”.



Equivalence class

Note

The relation “is strongly connected with” is not symmetric, and does not represent an equivalence class.

Trees

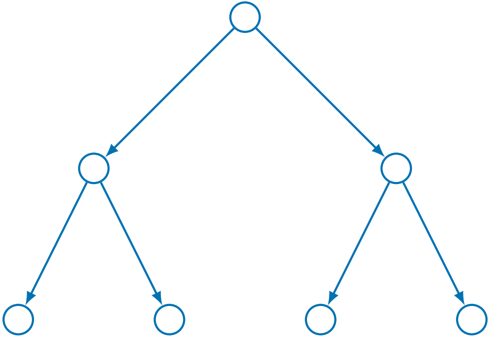
Motivation

- We introduce a family of graphs called "trees".
- They are useful in many applications.

Definition

Tree

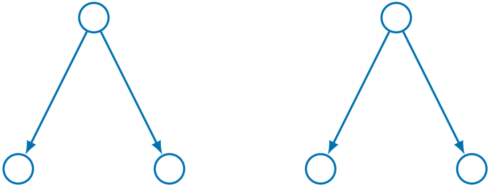
- Connected graph,
- without cycle.



Definition

Not a tree

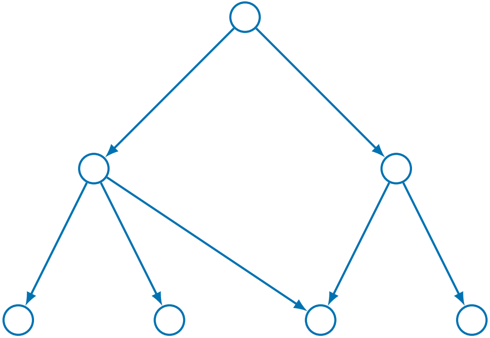
Not connected. Sometimes called a forest, as each connected component is a tree.



Definition

Not a tree

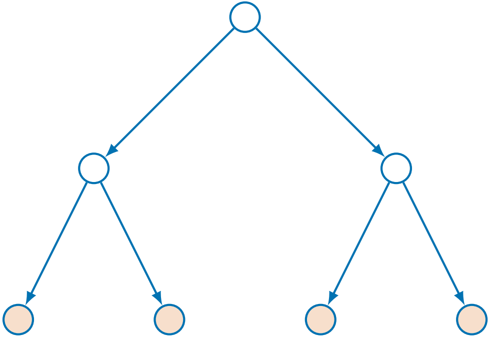
Contains a cycle.



Definition

Leaf

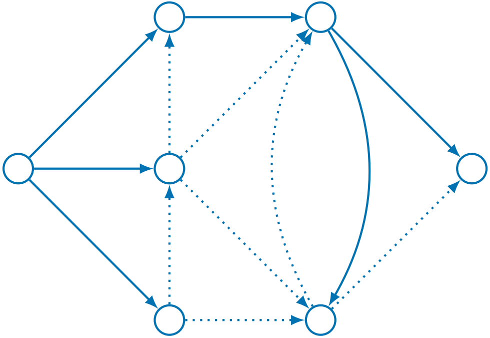
Node of degree 1.



Spanning tree

Definition

- Consider the graph $(\mathcal{V}, \mathcal{E}, \phi)$.
- The subgraph $(\mathcal{V}, \mathcal{E}', \phi')$
- is a spanning tree of $(\mathcal{V}, \mathcal{E}, \phi)$,
- if it is a tree.



Properties of trees

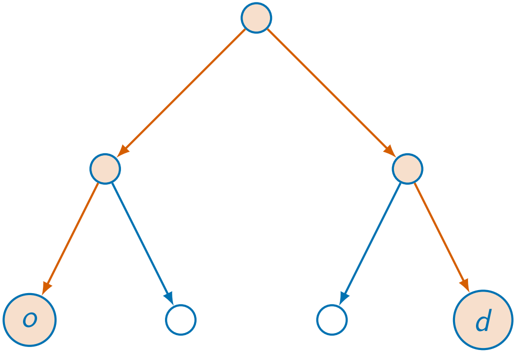
Motivation

- Trees have some interesting properties
- We review some of them.
- We also provide some characterizations of tree, involving these properties.

Lemma 21.9

A tree with at least one arc has at least two leafs.

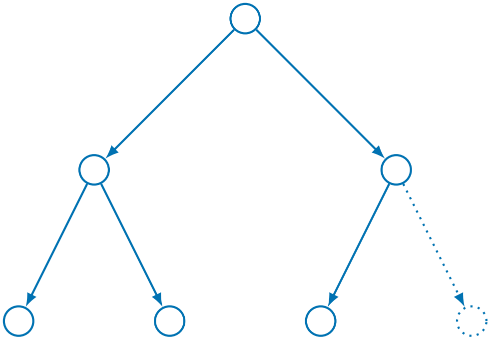
- Path $\mathcal{P}$ with maximum number of arcs.
- First node: o, last node d.
- As there is no cycle, $o \neq d$.
- Degree of $o \geq 1$.
- If degree of $o > 1$, another arc can make the path longer. Impossible.
- Degree of $o = 1$. It is a leaf.
- Same argument for d.



Number of nodes

A tree with n arcs has $m = n + 1$ nodes.

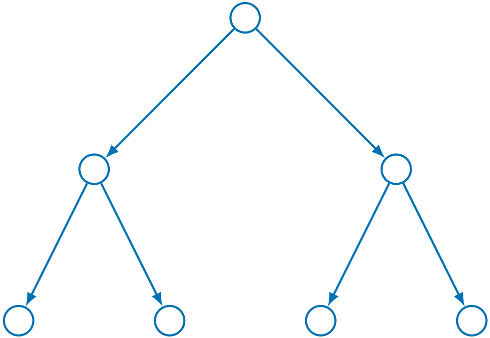
- Obvious for $n = 1$.
- Assume true for $n = p - 1$ arcs: there are $m = p$ nodes.
- Proof for a tree with $n = p$ arcs.
- Consider a leaf.
- Remove one node (the leaf) and one arc (the incident arc).
- We obtain a tree with $p - 1$ arcs.
- It has p nodes.
- The original tree has $p + 1$ nodes.



Single path

In a tree, there is exactly one path between any two nodes.

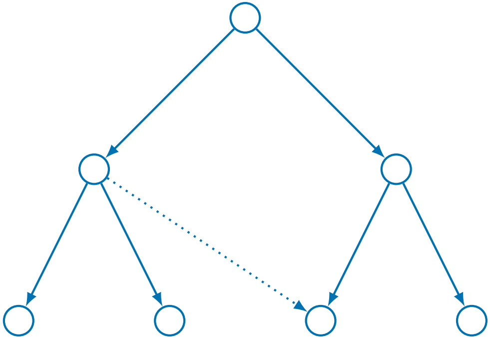
- A tree is connected, so there is at least one path.
- Suppose there are two different paths.
- They form a cycle.
- Impossible in a tree.



Cycle formation

In a tree, adding any arc forms a cycle.

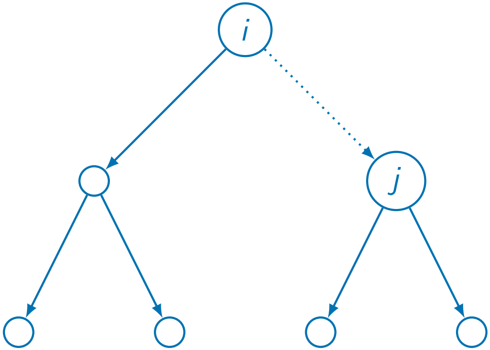
- Consider adding arc (i, j).
- A tree is connected, so there is a path connecting i and j.
- The added arc forms a cycle with the path.



Disconnection

In a tree, removing any arc disconnects the graph.

- Consider arc (i, j).
- There is a unique path from i to j.
- It is the link!
- Removing it disconnects i from j.



Characterization

Consider $G = (\mathcal{N}, \mathcal{A}, \phi)$ a directed graph with m nodes and n arcs. The following statements are all equivalent.

- G is a tree;
- G is connected and without cycles;
- There is a unique simple path connecting any two nodes;
- G has no cycle, and a simple cycle is formed if any arc is added;
- G is connected and the removal of any single arc disconnects the graph;
- G is connected and $n = m - 1$;
- G has no simple cycle and $n = m - 1$.

Flows

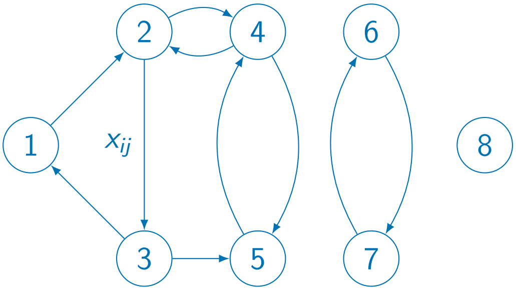
Motivation

- Physical networks are often used to transport objects or information: water, electricity, cars, internet packets, etc.
- We provide a generic representation of flows of these objects, and associate them with the graph.
- In our mathematical formalism, a directed graph is called a network when its nodes and arcs are associated with quantities.
- The first of the quantities is the flow.

Definition

$$x_{ij} \in \mathbb{R}$$

- Amount of "things" traversing the arc during a given time period.
- Associated with each arc (i, j).
- Units are arbitrary and context dependent.
- Time period is irrelevant and long enough.
- The sign corresponds to the direction.



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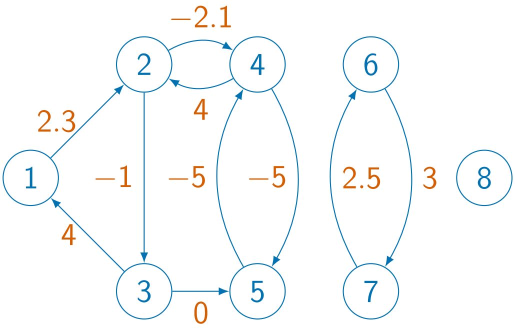


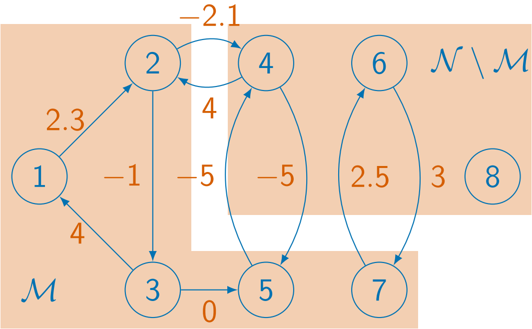
Flow through a cut

$$X(\Gamma) = \sum_{(i,j)\in\Gamma^{\rightarrow}} x_{ij} - \sum_{(i,j)\in\Gamma^{\leftarrow}} x_{ij},$$

$\sum_{(i,j)\in\Gamma^{\rightarrow}} x_{ij} =$
$x_{24} + x_{54} + x_{76} = -2.1 - 5 + 2.5 = -4.6$

$\sum_{(i,j)\in\Gamma^{\leftarrow}} x_{ij} =$
$x_{42} + x_{45} + x_{67} = 4 - 5 + 3 = 2.$
Total flow through the cut:

$$-4.6 - 2 = -6.6.$$



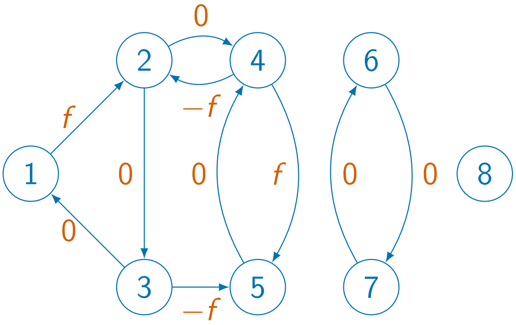
Simple path flow

$x \in \mathbb{R}^n$ such that

$$x_{ij} = \begin{cases} f & \text{if } (i,j) \in P^{\rightarrow} \\ -f & \text{if } (i,j) \in P^{\leftarrow} \\ 0 & \text{otherwise.} \end{cases}$$

Example

$$1 \rightarrow 2 \leftarrow 4 \rightarrow 5 \leftarrow 3$$



Capacities

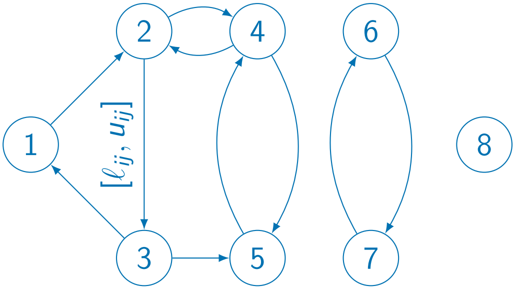
Motivation

- There is a limit to the quantity of flow that can be transported on each section of a physical network.
- For instance, the quantity of water flowing through a pipe depends on the section area.
- One lane of a highway cannot accommodate more than 2400 veh/h.
- This limit is called the capacity.
- In our mathematical formalism, we may impose bounds on flows.

Definition

$$\ell_{ij} \leq x_{ij} \leq u_{ij}.$$

- $\ell_{ij} \in \mathbb{R}$: minimum quantity of flow.
- $u_{ij} \in \mathbb{R}$: maximum quantity of flow.
- Associated with each arc (i, j).
- Units are the same as x_{ij}.
- In practice, we often have:
  - $\ell_{ij} = 0$, or,
  - $\ell_{ij} = -u_{ij}$.



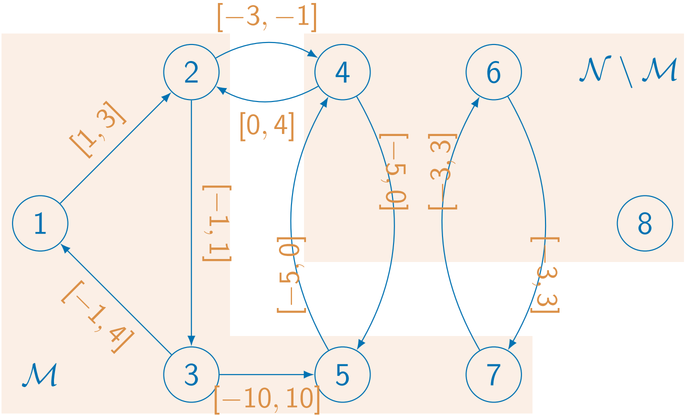
Capacity of a cut

$$U(\Gamma) = \sum_{(i,j)\in\Gamma^{\rightarrow}} u_{ij} - \sum_{(i,j)\in\Gamma^{\leftarrow}} \ell_{ij}.$$

$\sum_{(i,j)\in\Gamma^{\rightarrow}} u_{ij} =$
$u_{24} + u_{54} + u_{76} = -1 + 0 + 3 = 2$

$\sum_{(i,j)\in\Gamma^{\leftarrow}} \ell_{ij} =$
$\ell_{42} + \ell_{45} + \ell_{67} = 0 - 5 - 3 = -8$

$U(\Gamma) =$ 2-(-8)=10



Capacity of a cut

Upper bound on the flow

$$X(\Gamma) \leq U(\Gamma).$$

Saturated cut

$$X(\Gamma) = U(\Gamma).$$

Supply and demand

Motivation

- Nodes can also be associated with quantities.
- For instance, the supply is a quantity of flow that a node is injecting on a network.
- In logistics, a warehouse is supplying flow of goods on the network.
- The demand is a quantity of flow absorbed by a node.
- In logistics, a customer is collecting the flow from the network.
- We characterizes these notions in our mathematical formalism.

Divergence

Flow leaving a node

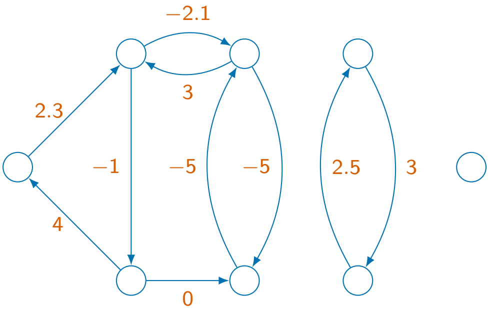
$$\sum_{j|(i,j)\in\mathcal{A}} x_{ij}.$$

-2.1-1=-3.1

Flow entering a node

$$\sum_{k|(k,i)\in\mathcal{A}} x_{ki}.$$

2.3+3=5.3

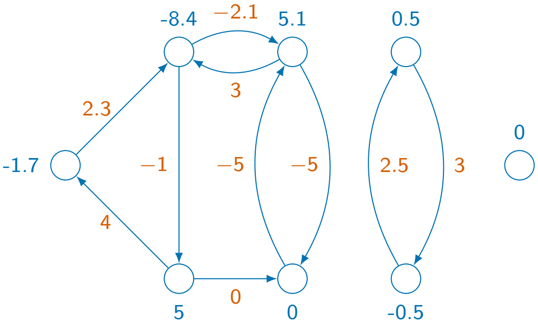


Divergence

Divergence

$$\mathrm{div}(x)_i = \sum_{j|(i,j)\in\mathcal{A}} x_{ij} - \sum_{k|(k,i)\in\mathcal{A}} x_{ki}.$$

-3.1-5.3=-8.4



Supply and demand

Supply node

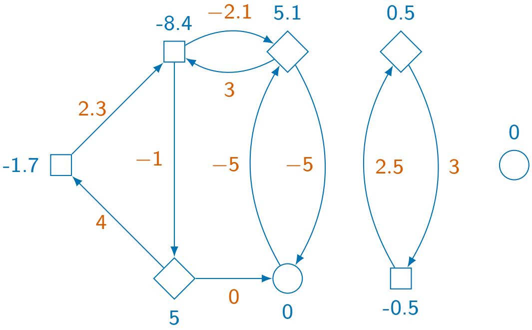
$$\mathrm{div}(x)_i > 0.$$

Demand node

$$\mathrm{div}(x)_i < 0.$$

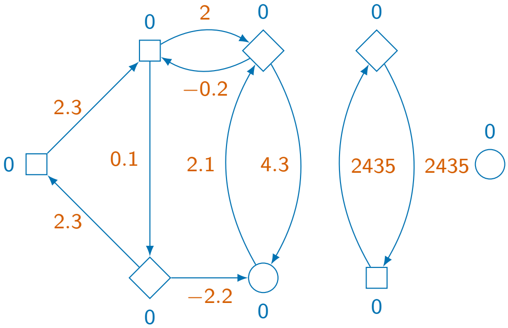
In any case

$$\sum_{i \in \mathcal{N}} \mathrm{div}(x)_i = 0.$$



Circulation

$$\text{div}(x)_i = 0, \ \forall i \in \mathcal{N}.$$



Costs

Motivation

- Moving flow along an arc generate costs.
- In our formalism, we consider proportional costs.
- Modeling costs is not necessarily easy.
- We discuss two common issues.

Costs

$$c_{ij}, \forall (i,j) \in \mathcal{A}.$$

Data of the problem

- Cost of transporting one unit of flow on arc (i,j).
- Unit is arbitrary.
- Total cost for arc (i,j):

$$c_{ij} x_{ij}.$$

- Total cost for the network:

$$\sum_{(i,j)\in\mathcal{A}} c_{ij} x_{ij}.$$

Generalized cost

Toll road

- travel time (say 30 minutes),
- travel cost (say 10 CHF).

Value of time

30 CHF/hour or 0.5 CHF/min.

Generalized cost

- In CHF: $30 \text{ min} \times 0.5 \text{ CHF/min} + 10 \text{ CHF} = 25 \text{ CHF}$.
- In min: $30 \text{ min.} + 10 \text{ CHF} / (0.5 \text{ CHF/min}) = 50 \text{ min}$.

Link additivity

Cost of a path

$$C(P) = \sum_{(i,j)\in P^{\rightarrow}} c_{ij}x_{ij} - \sum_{(i,j)\in P^{\leftarrow}} c_{ij}x_{ij}.$$

Cost along a simple path flow

$$\begin{aligned} C(P) &= \sum_{(i,j)\in P^{\rightarrow}} fc_{ij} - \sum_{(i,j)\in P^{\leftarrow}} fc_{ij} \\ &= f\Big(\sum_{(i,j)\in P^{\rightarrow}} c_{ij} - \sum_{(i,j)\in P^{\leftarrow}} c_{ij}\Big). \end{aligned}$$

Example

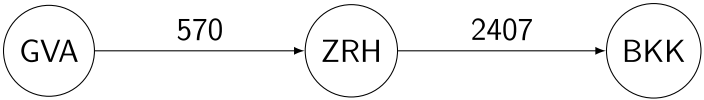
GVA $\rightarrow$ ZRH $\rightarrow$ BKK
CHF 2412

GVA $\rightarrow$ ZRH
CHF 570

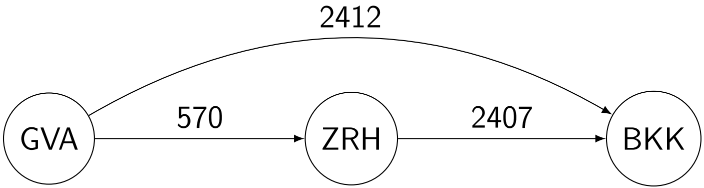
ZRH $\rightarrow$ BKK
CHF 2407

Modeling

First model



Second model

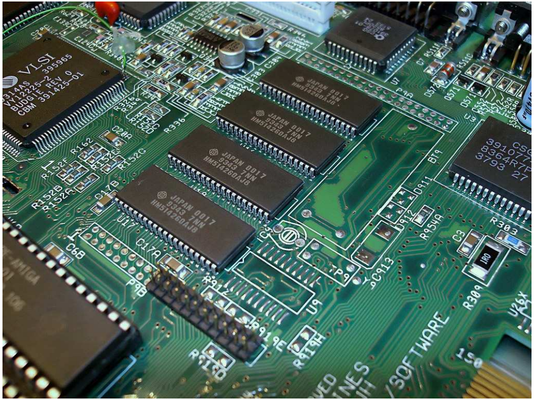


Computer representation

Motivation

- We, humans, usually use maps or schematics to look at networks.
- We obtain an overview of the overall topology.
- But computers do not have this bird eyes's view.
- We introduce here two possible representations of networks in a computer.

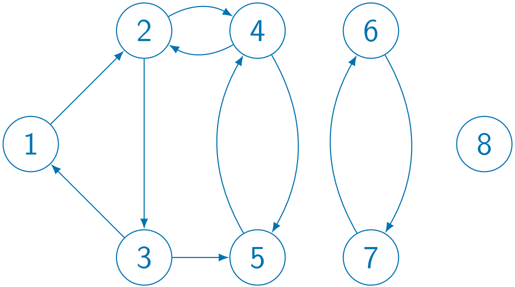
Computer representation



Adjacency matrix

$$A(i,j) = \begin{cases} 1 \text{ if } (i,j) \in \mathcal{A}, \\ 0 \text{ otherwise.} \end{cases}$$

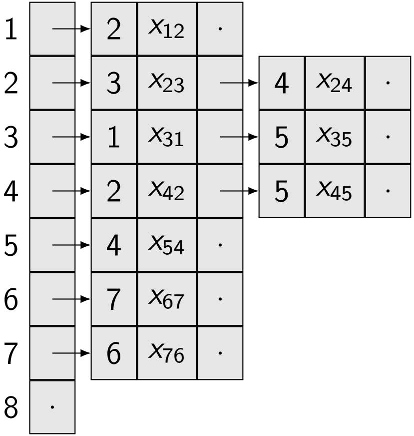
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

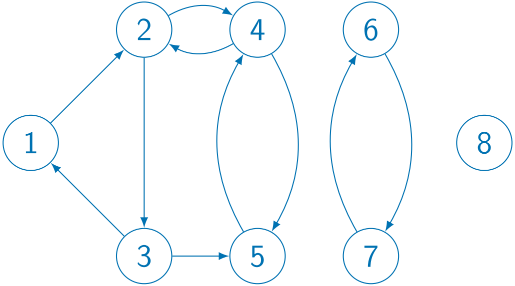


Adjacency matrix

- Valid representation because the incidence function is injective.
- Arc numbering convention for storing arc quantities.
- Sparse matrix: efficient storage techniques should be used.
- For instance, adjacency lists.

Adjacency lists





Summary

- Graphs and subgraphs.
- Cuts.
- Paths and connected components.
- Trees.
- Flows and capacity.
- Supply and demand.
- Costs.
- Computer representations.