+62/1/40+

Teacher: Michel Bierlaire
Course: Introduction to optimization and operations research

Tuesday 30 January 2024 (09:15 - 12:15)

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Part 2: open questions

SCIPER: **375367** Signature:

Wait until the beginning of the exam before turning the page. This document is printed double-sided and has 4 open questions. Do not remove the staple. This part of the exam contains 20 pages.

Carefully read the following instructions:

- Place your student card on the table.
- You are allowed to have a **4 pages** (8 sides) **handwritten** summary. Any summary which is not handwritten cannot be used and **will be confiscated**. The summary must clearly show your name and today's date. The summary will be collected at the end of the exam.
- Draft paper is available, so **please write cleanly on your exam**. Calculators and electronic devices are **not allowed**.
- If a question contains a mistake, the teacher can remove it from the exam.

Open questions (56 points)

Linear optimization (14 points)

Consider the following linear optimization problem:

$$\begin{aligned} \min \quad & 2x_1 - 5x_2, \\ \text{s.t.:} \quad & x_1 - 3x_2 \geq -9, && (245) \\ & x_1 + x_2 \leq 7, && (246) \\ & x_1 \geq 0, && (247) \\ & x_2 \geq 0. && (248) \end{aligned}$$

(a) (**2 points**) Consider the solution of the original problem where $x_1 = 7$ and $x_2 = 0$. Does the solution $x_1 = 7$, $x_2 = 0$ verify all the constraints of the original problem? Which constraints are active at (x_1, x_2)? Justify your answer.

(b) (**2.75 points**) Represent the feasible region graphically. Solve the problem using the graphical method. Include as well one level line, as well as the direction of steepest descent. Provide the values of the two variables at the optimal solution, and represent it on the graph. Report the value of the objective function for the optimal solution.

Hint: You can use the grid below to represent your graph.

(c) (**3.5 points**) Write the problem in standard form. Give explicitly the matrix $\boldsymbol{A}$ and vectors $\boldsymbol{b}$ and $\boldsymbol{c}$.

(d) Consider the problem in standard form. Consider the case where the variables x_1 and x_2 are in the basis.
1. (**3.5 points**) Check if it is a valid basis. Check if the basic solution is feasible. If so, identify the corresponding vertex on the polyhedron and explain if it is degenerate or not. If not, identify which of the 4 constraints are violated.

Hints:

- *The inverse of matrix* $M = \begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix}$ *is matrix* $M^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{-1}{4} & \frac{1}{4} \end{pmatrix}$.
- *The inverse of matrix* $M = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ *is matrix* $M^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.
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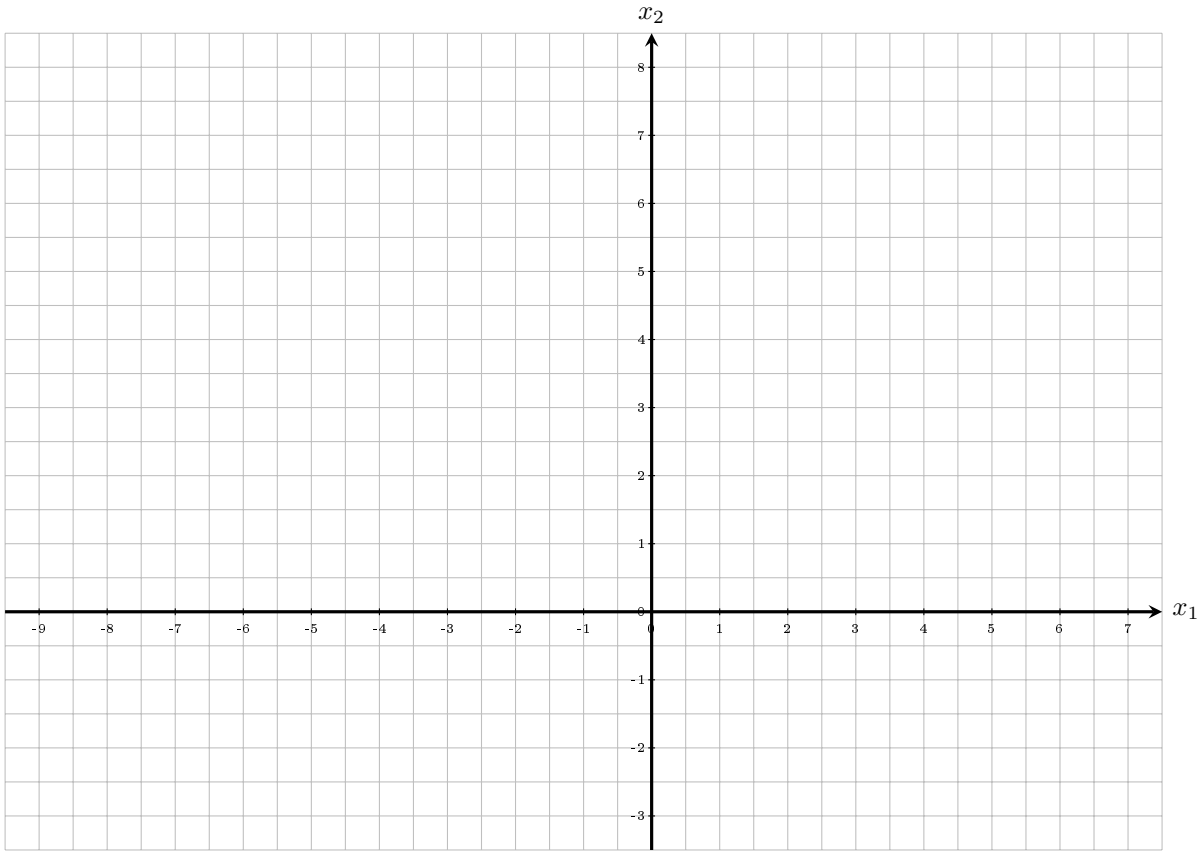
- *The inverse of matrix* $M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ *is matrix* $M^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

2. (**2.25 points**) Calculate the basic direction corresponding to the non-basic variable x_3. Is it feasible? Represent it on the graph, and justify algebraically your answer.

Answer for question 1:



Question 2: Duality (14 points)

Consider the following linear optimization problem:

$$\min_{x \in \mathbb{R}^2} -3x_1 + 2x_2 + 7x_3$$

subject to

$$\begin{aligned}
-x_1 \quad -x_2 \quad\quad\quad &\leq 4, \\
x_1 \quad\quad\quad +3x_3 \quad &\geq -5, \\
7x_1 \quad -2x_2 \quad +x_3 \quad &= 6, \\
x_1 \quad\quad\quad\quad\quad &\leq 0, \\
\quad\quad x_2 \quad\quad\quad &\geq 0, \\
\quad\quad\quad\quad x_3 \quad &\geq -5.
\end{aligned}$$

(a) (**10 points**) Calculate the dual of the linear optimization problem using the Lagrangian function. *Hint:* Make sure to write the Lagrangian function, then the constraints, and eliminate any unnecessary dual variables not present in the dual objective function to obtain the dual problem.

(b) (**4 points**) Let $x^* = (0, 0, 6)$ be the optimal solution of the primal. Use complimentarity slackness to solve the dual problem. Make sure to clearly write down the equations that connect the primal variables with the dual variables (complimentarity slackness) and how you compute the dual variables.

Answer for question 2

Question 3: Shortest path (15 points)

Eight students from the MATH course have chosen to connect on a new social network platform for course-related discussions. On this platform, only one-on-one message exchanges (private chats) are possible. The social distance among these students, a measure of how close people are, is provided in the table below. It is important to note that pairs of students not mentioned in this table do not interact.

From	To	Social distance [Avg. # messages per day^{-1}]
Aya	Chloé	1/2
Aya	Elena	1/10
Chloé	Aya	1/2
Chloé	Isha	1/8
Chloé	Liu	1/4
Elena	Aya	1/7
Elena	Chloé	1/8
Elena	Liu	1/5
Elena	Priyank	1/2
Isha	Chloé	1/8
Isha	Liu	1/2
Isha	Priyank	1/10
Isha	Youssouf	1/8
Liu	Elena	1/6
Liu	Priyank	1
Priyank	Liu	1/2
Priyank	Isha	1/7
Priyank	Youssouf	1/3
Youssouf	Isha	1/8
Youssouf	Priyank	1

An ill-intentioned individual, Aya, aims to disseminate false information among group members with the intention of lowering their colleagues' marks on the final exam. Aya seeks to identify the student who will be most challenging to influence and determine those with the most significant impact (i.e., critical for information dissemination). To do that, Aya needs to calculate the "shortest" path to each classmate. To accomplish this,

(a) (**4 points**) Complete the corresponding shortest path network. Use the node setup provided on the next page. You must give the arcs and their corresponding cost.

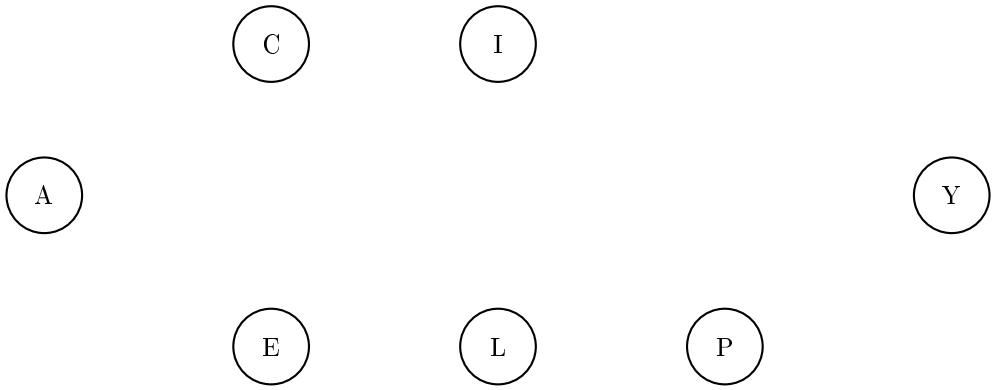
(b) (**7 points**) By using Dijkstra's algorithm, find the smallest social distance between Aya and all other members. Use the table provided on the next page. Report the labels of all the nodes at each iteration. You must express the labels using fractional values.

(c) (**3 points**) Draw the shortest path spanning tree from Aya. Use the node setup provided on the next page. You must give the arcs and the labels of each node.

(d) (**1 points**) The betweenness centrality is a metric that counts the number of shortest paths that pass through a node. It is used to identify the most influential members in a social network. Give the betweenness centrality scores for all the students and identify the most influential member.

Answer for question 3

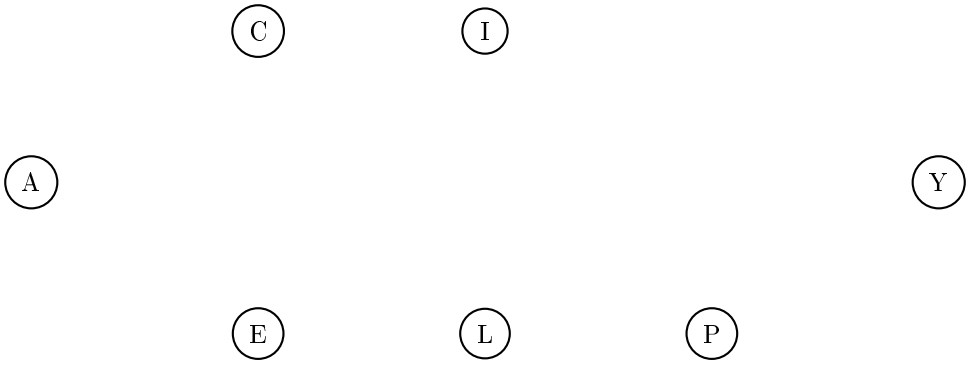
a) Use this node setup to answer the question. The student IDs are as follows: Aya (A), Chloé (C), Elena (E), Isha (I), Lui (L), Priyank (P), and Youssef (Y).



b) Use this table to answer the question.

it.	S	i	λ_A	λ_C	λ_E	λ_I	λ_L	λ_P	λ_Y
0	{A}	A	0	∞	∞	∞	∞	∞	∞
1									
2									
3									
4									
5									
6									
7									
8									
9									
10									
11									
12									

c) Use this node setup to answer the question.



d) Use this table to answer the question.

Student	**Betweenness centrality**
Chloé	
Elena	
Isha	
Liu	
Priyank	
Youssouf	

The most influential member of the group (i.e., the one with the largest betweenness centrality) is:

Question 4: Non Linear Optimization (13 points)

Consider the following optimization problem:

$$\min_{x \in \mathbb{R}^2} f(x) = 2x_1^2 + 2x_1x_2 + 2x_2^2 - 6x_2$$

(a) (**2.5 points**) Show that the function has a global minimum and calculate it. Justify your answer.
Hint: The formula for calculating eigenvalues is:

$$\det(\nabla^2 f(x^*) - \lambda \mathcal{I}_2)$$

(b) (**4 points**) Perform one iteration of the local Newton method starting from $x_0 = (0,0)^T$. Comment on the obtained solution and convergence of Newton's method.

(c) (**6.5 points**) Using the two Wolfe conditions, calculate the bounds on the step α_0 in the Newton direction at the point $x_0 = (0,0)^T$. The parameters to be used for Wolfe's conditions are: $\beta_1 = 0.1$ and $\beta_2 = 0.9$. Justify your answer.
Hint: The first Wolfe condition can be calculated as follows:

$$f(x_k + \alpha_k d_k) \leq f(x_k) + \alpha_k \beta_1 \nabla f(x_k)^T d_k$$

The second Wolfe condition can be calculated as follows:

$$\frac{\nabla f(x_k + \alpha_k d_k)^T d_k}{\nabla f(x_k)^T d_k} \leq \beta_2$$

Answer for question 4