Introduction to optimization and operations research
Interactive session

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The correct answer is identified by the bold font.

1 Linear optimization

1.1 Degeneracy

1.2 Definition

Consider a linear optimization problem with n variables and m constraints. A feasible basic solution is degenerate if

1. **more than n constraints are active at x.**
2. more than m constraints are active at x.
3. less than n constraints are actives at x.
4. less than m constraints are actives at x.

1.3 Example

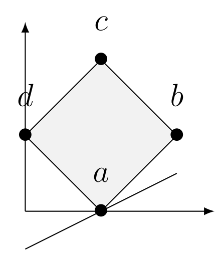
Consider the polyhedron:

$$\mathcal{P} = \{(x_1, x_2) \in \mathbb{R}^2 \text{ s. t.}$$

$$
\begin{aligned}
x_1 - x_2 &\leq 1, \\
x_1 + x_2 &\leq 3, \\
x_1 - x_2 &\geq -1, \\
x_1 + x_2 &\geq 1, \\
x_1 - 2x_2 &\leq 1, \\
x_1 &\geq 0, \\
x_2 &\geq 0.
\end{aligned}
$$

$$\}$$

represented in the following figure.



How many vertices correspond to degenerate feasible basic solutions? 0, 1, **2**, 3, 4.

It is a problem with $n = 2$ variables. Therefore, if there are more than 2 active constraints at a vertex, we are facing degeneracy.

- At vertex a, there are 4 active constraints. Therefore, the corresponding feasible basic solution is degenerate.
- At vertex b, there are 2 active constraints. Therefore, the corresponding feasible basic solution is not degenerate.
- At vertex c, there are 2 active constraints. Therefore, the corresponding feasible basic solution is not degenerate.
- At vertex d, there are 3 active constraints. Therefore, the corresponding feasible basic solution is degenerate.

Therefore, there are two vertices with degeneracy.

1.4 Step along the direction

1. The length of the step α to perform along the basic direction d_p is determined using exactly one of the following conditions. Which one?

   (a) The gradient at $x+\alpha d_p$ is zero. No, the gradient is constant and always equal to c. Therefore, except for a trivial problem where $c = 0$, the gradient is never zero.

   (b) $\alpha = 1$. No, the value one is arbitrary.

   (c) α is equal to the reduced cost associated with d_p. No, the reduced cost is the slope, the directional derivative, of the objective function in the direction d_p.

   (d) α **corresponds to the first constraint that is activated.** Yes, we follow the descent direction d_p as far as we can, which is when the first constraint is met. Beyond, the iterate would become infeasible.

   (e) α corresponds to the last constraint that is activated. No, it must be the first, as discussed above.

   (f) There are $m+1$ variables in the basis at $x+\alpha d_p$. No, there are always m variables in the basis.

   (g) There are $m-1$ variables in the basis at $x+\alpha d_p$. No, there are always m variables in the basis.

2. Consider an optimal feasible basic solution, such that the reduced cost associated with the non basic variable x_j is $\bar{c}_j < 0$. What is the length α of the step to perform along the basic direction d_j?

   (a) **0**,

   (b) 1,

   (c) $+\infty$,

   (d) this situation is not possible.

   The situation is possible, when the optimal basis is degenerate. As the solution is optimal, any descent direction must be infeasible. And d_j is a descent direction, as the reduced cost is negative. Therefore, the step is zero.

2 Duality

2.1 Lagrangian

Consider a linear optimization problem in standard form, with n non negative variables, and m equality constraints. x^* is an optimal basic solution, associated with the basis B. Consider as well the dual problem. Which one of those statements is **wrong**?

1. The optimal value of the dual problem is the same as the optimal value of the primal. True. This is the strong duality theorem.

2. **The optimal solution of the dual is the same as the optimal solution of the primal.** No. The two problems have a different solution space. The primal problem has n variables, while the dual problem has m variables.

3. The dual problem involves m variables and n inequality constraints. Yes, indeed. Note that the dual variables associated with the non negativity constraints of the primal can always be eliminated.

4. If $x_i^* > 0$, the corresponding constraint of the dual is active at the solution. Indeed, due to the complementarity slackness conditions.

3 Networks

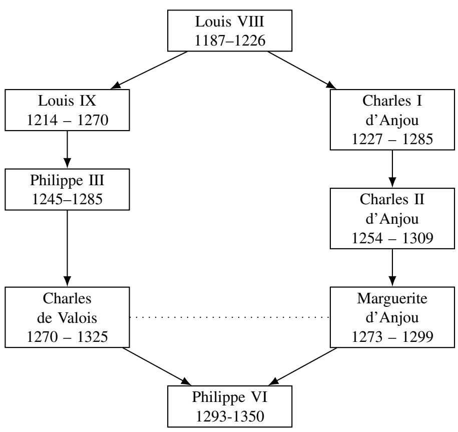
3.1 Trees

1. Consider a tree with m nodes and n arcs. Which one among these statements is **wrong**?

    - **The shortest path between two leaves has a least two arcs.** Wrong. A graph with one arc and two nodes is a tree, and there is only one arc in the path between the two nodes.
    - $n = m - 1$. Correct. See Theorem 21.10.
    - Adding one arc always creates a cycle. Correct. See Theorem 21.10.
    - If $n \geq 1$, there are at least two leaves. Correct. See Lemma 21.9.
    - Removing one arc disconnects the graph. Correct. See Theorem 21.10.
    - There is at least one path between each pair of nodes. Correct, as a tree is connected. Actually, Theorem 21.10 states that there is a unique simple path connecting any two nodes.

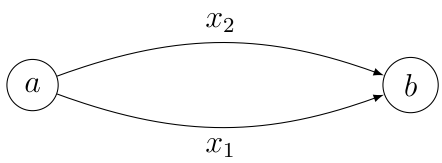
2. A "family tree" is a graph where the nodes correspond to individuals, and the arcs correspond to the relationship "to be the parent of". Is it a tree? No. If two persons with a common ancestor have a child, it forms a cycle in the network. Indeed, the child is connected with the common ancestor with two paths: one through her father, and one through her mother. Therefore, the network representing a "family tree" is not a tree.

    For example, considering the family tree of the kings of France, there is a cycle through Philippe VI Valois (1293-1350) and Louis VIII (1187–1226), as illustrated below. Louis VIII is the great-grandfather of both parents of Philippe VI Valois.



3.2 Circulation

Consider the following network with two nodes and two arcs, and the vector of flow $x \in \mathbb{R}^2$:



Denote y_a and y_b the divergence at nodes a and b, respectively. What condition is sufficient for the vector of flow to be a circulation?

1. It is not possible to have a circulation in this network. No, it is possible. As explained below, any flow such that $x_1 = -x_2$ is a circulation.

2. $y_a = x_1 + x_2$. No. This is just the definition of the divergence, valid for any vector of flow.

3. $y_a + y_b = 0$. No. This condition is verified for any vector of flow.

4. $x_1 = x_2$. No. If $x_1 = x_2 = 1$, node a is a supply node with $y_a = 2$, and node b is a demand node with $y_b = -2$. This is not a circulation.

5. $x_1 + x_2 = 0$. Yes. Indeed, the divergences are

$$
\begin{aligned}
y_a &= x_1 + x_2 = 0, \\
y_b &= -x_1 - x_2 = 0.
\end{aligned}
$$

   All divergences are zero. We have a circulation.

4 Transhipment

4.1 Integral solutions

The simplex algorithm applied to a transhipment problem always generates an integral solution. True/**False**.

This is false. It is true when the supply data of the problem are integer. But if it is not the case, the optimal solution may not be integral, as in the simple example below.