Numerical Analysis
GC / SIE
Ordinary Differential Equations

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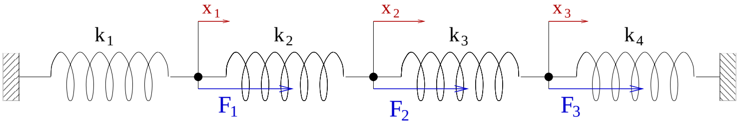
A mechanical example

Let us consider the springs configuration below under applied forces.
We want to compute the elongation of every spring.



Forces balance in each node

Linear system with 3 equations and 3 unknowns: $K\mathbf{x} = \mathbf{F}$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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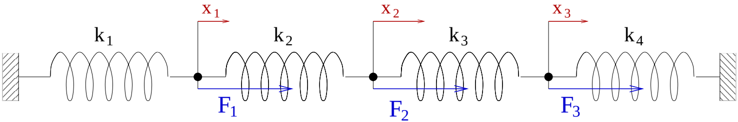

Forces balance in each node

$$\begin{cases} k_1 x_1 - k_2(x_2 - x_1) = F_1 \\ k_2(x_2 - x_1) - k_3(x_3 - x_2) = F_2 \\ k_3(x_3 - x_2) + k_4 x_3 = F_3 \end{cases}$$

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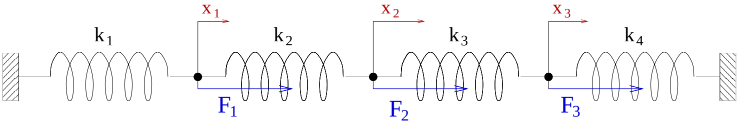

Forces balance in each node

$$\Rightarrow \quad \begin{cases} (k_1 + k_2)x_1 & - & k_2 x_2 & & & = F_1 \\ -k_2 x_1 & + & (k_2 + k_3) x_2 & - & k_3 x_3 & = F_2 \\ & - & k_3 x_2 & + & (k_3 + k_4) x_3 & = F_3 \end{cases}$$

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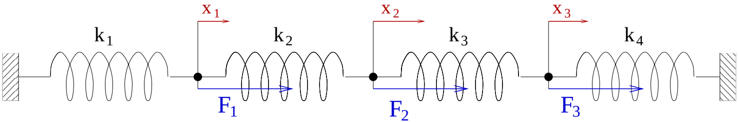

Forces balance in each node

$$\Rightarrow \quad \begin{cases} (k_1 + k_2)x_1 - k_2 x_2 = F_1 \\ -k_2 x_1 + (k_2 + k_3)x_2 - k_3 x_3 = F_2 \\ - k_3 x_2 + (k_3 + k_4)x_3 = F_3 \end{cases}$$

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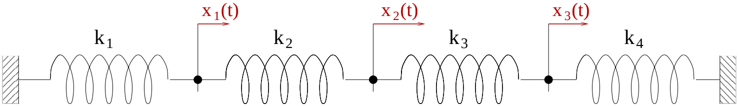
A mechanical example: dynamic case

Let us assume that the nodes x_1, x_2, x_3 have a mass m and that there are no external forces:



Newtons's law (F=ma)

$$\begin{cases} -(k_1+k_2)x_1 + k_2 x_2 = m\ddot{x}_1 \\ k_2 x_1 - (k_2+k_3)x_2 + k_3 x_3 = m\ddot{x}_2 \\ k_3 x_2 - (k_3+k_4)x_3 = m\ddot{x}_3 \end{cases}$$

Second order ordinary differential equations system with 3 equations:

$$\ddot{\mathbf{x}} = -\frac{1}{m} K \mathbf{x}$$

A mechanical example: dynamic case

We introduce the velocity variables $v_1 = \dot{x}_1$, $v_2 = \dot{x}_2$, $v_3 = \dot{x}_3$, the precedent system becomes a system with 6 first order ordinary differential equations:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \frac{1}{m}\left(-(k_1 + k_2)x_1 + k_2 x_2\right) \\ \frac{1}{m}\left(k_2 x_1 - (k_2 + k_3)x_2 + k_3 x_3\right) \\ \frac{1}{m}\left(k_3 x_2 - (k_3 + k_4)x_3\right) \end{bmatrix}$$

that can be written in compact form

$$\dot{\mathbf{u}}(t) = \mathbf{f}(t, \mathbf{u}(t))$$

where $\mathbf{u}(t) = [x_1(t), x_2(t), x_3(t), v_1(t), v_2(t), v_3(t)]^T$.

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A more complex application

By imposing Newton's law for each rod we get a large system of ordinary differential equations to be solved. We definitely need a computer!