PROBABILITY AND STATISTICS (MATH-235)

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Mock exam - questions

Date: 18th of December, 2024
Time: 13:15–15:15

Name: ________________________

SCIPER: ________________________

INSTRUCTIONS TO CANDIDATES

- This is a **mock exam**. To obtain the maximum number of points you should be clear about your reasoning and present your arguments explicitly. You have **2 hours** to complete the **mock exam**, however for the **final exam** you will have **3 hours**. Accordingly, the number of questions will increase in the **final exam**.
- All that can be used for this exam is a pen and the cheat sheet that **we provide for you**. No external notes, books, summaries, formula collections or calculators are allowed. All questions should be answered.
- The finest enumerated item in each question will be marked on a scale of $0-2$ points, indicating an incorrect, partially correct and completely correct answer respectively (half-points are not given). **The exam has 6 questions with a total of 54 points**.
- ~~**Write the answer to every question in the other booklet (final exam - answers)**~~ (*We do not have an answer booklet for the mock exam, one will be given at the final exam*). Scrap paper will be provided for rough work, but only answers written in the booklet will be marked.
- At the end of the exam, you will have to return everything : the booklet with the questions, the booklet with your answers, and the scrap paper.
- When the exercise directs you to "**Simplify!**" provide a simple fraction, a decimal, or the simplest form possible using the function or parameter in terms of which your answer must be given. If you are explicitly instructed with "**Do not simplify!**", it is acceptable to leave your answer as an integral, sum, or another more complex form, depending on the context.

Mark question 1 (TOT: 8 points):

Mark question 2 (TOT: 10 points):

Mark question 3 (TOT: 6 points):

Mark question 4 (TOT: 12 points):

Mark question 5 (TOT: 6 points):

Mark question 6 (TOT: 12 points):

Variable definitions

All the random variables have finite mean and variance.

Question 1.

Two coins are in a hat. The coins look alike, but one coin is fair (with probability 1/2 of Heads), while the other coin is biased, with probability 1/4 of Heads. One of the coins is randomly pulled from the hat, without knowing which of the two it is. Call the chosen coin "Coin C".

(a) Coin C is tossed twice, showing Heads both times. Given this information, what is the probability that Coin C is the fair coin? (Simplify.)

(b) Now suppose that coin C is tossed twice, showing Heads for the first time and Tails for the second. Given this information, what is the probability that Coin C is the fair coin? (Simplify.)

(c) Are the events "first toss of Coin C is Heads" and "second toss of Coin C is Heads" independent? Explain briefly.

(d) Find the probability that in 10 flips of Coin C, there will be exactly 3 Heads. (The coin is equally likely to be either of the 2 coins; do not assume it already landed Heads twice as in (a) or that it landed Heads then Tails as in (b). Do not simplify.)

Solution 1.

(a) By Bayes' Rule and the law of total probability,

$$P(\text{fair} \mid HH) = \frac{P(HH \mid \text{fair})P(\text{fair})}{P(HH)} = \frac{\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)}{\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{16}\right)\left(\frac{1}{2}\right)} = \frac{4}{5}.$$

(b) Similarly,

$$P(\text{fair} \mid HT) = \frac{P(HT \mid \text{fair})P(\text{fair})}{P(HT)} = \frac{\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)}{\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{3}{16}\right)\left(\frac{1}{2}\right)} = \frac{4}{7}.$$

(c) They're not independent: the first toss being Heads is evidence in favor of the coin being the fair coin, giving information about probabilities for the second toss. A formal calculation showing this also earns full marks.

(d) Let X be the number of Heads in 10 tosses. By the Law of Total Probability (conditioning on which of the two coins C is),

$$\begin{aligned} P(X = 3) &= P(X = 3 \mid \text{fair})P(\text{fair}) + P(X = 3 \mid \text{biased})P(\text{biased}), \\ &= \binom{10}{3}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right) + \binom{10}{3}\left(\frac{1}{4}\right)^{3}\left(\frac{3}{4}\right)^{7}\left(\frac{1}{2}\right), \\ &= \frac{1}{2}\binom{10}{3}\left(\frac{1}{2^{10}} + \frac{3^7}{4^{10}}\right). \end{aligned}$$

Question 2.

(a) Let $X_1, X_2, \ldots$ be independent $\mathcal{N}(0, 4)$ random variables, and let J be the smallest value of j such that $X_j > 4$ (i.e., the index of the first X_j exceeding 4).

(i) What is the distribution of $J - 1$? Derive the parameter(s) of it in terms of a probability, using the X_j-s!

(ii) In terms of the standard Normal CDF Φ, find $E(J)$ (simplify).

(b) Let f and g be PDFs with $f(x) > 0$ and $g(x) > 0$ for all x. Let X be a random variable with PDF f. Find the expected value of the ratio $\frac{g(X)}{f(X)}$ (simplify).

(c) Define $F(x) = e^{-e^{-x}}$. This is a CDF (called the Gumbel distribution) and is a continuous, strictly increasing function. Let X have CDF F, and define $W = F(X)$.

(i) Find the CDF of W.

(ii) What are the mean and variance of W (simplify)?

Solution 2.

(a) (i) We have $J - 1 \sim \text{Geom}(p)$ with $p = P(X_1 > 4) = P(X_1/2 > 2) = 1 - \Phi(2)$, as J counts the number of tosses needed to have success, so it has a "First Success" distribution. The probability of success is determined by the distribution of the X-s, that is, we only have a success if $P(X_i > 4) = P(X_i/2 > 2)$, where $X_i/2$ has a standard normal distribution by rescaling.

(ii) As $J - 1 \sim \text{Geom}(p)$, the expectation of $J - 1$ is equal to $\frac{1-p}{p}$, thus $E(J) = 1 + \frac{1-p}{p} = \frac{1}{p}$, where $p = 1 - \Phi(2)$. So in summary $E(J) = \frac{1}{1-\Phi(2)}$.

(b) By LOTUS,

$$E\left(\frac{g(X)}{f(X)}\right) = \int_{-\infty}^{\infty} \frac{g(x)}{f(x)} f(x)\, dx = \int_{-\infty}^{\infty} g(x)\, dx = 1.$$

(c) (i) Note that W is obtained by plugging X into its own CDF. The CDF of W is

$$P(W \leq w) = P(F(X) \leq w) = P(X \leq F^{-1}(w)) = F(F^{-1}(w)) = w,$$

for $0 < w < 1$, so $W \sim \text{Unif}(0, 1)$.

(ii) Since $W \sim \text{Unif}(0, 1)$

$$E(W) = \frac{1}{2} \quad \text{and} \quad \text{Var}(W) = \frac{1}{12}.$$

Question 3.

Let $U \sim \text{Unif}(0,1)$, and $X = \ln\left(\frac{U}{1-U}\right)$.

(a) Write down (but do not compute) an integral giving $E(X^2)$.

(b) Find the CDF of X (simplify).

(c) Find $E(X)$ *without using calculus* (simplify).
Hint: $1-U$ has the same distribution as U.

Solution 3.

(a) By LOTUS,

$$E(X^2) = \int_0^1 \left(\ln\left(\frac{u}{1-u}\right)\right)^2 du.$$

(b) This can be done directly or by Universality of the Uniform. For the latter, solve

$$x = \ln\left(\frac{u}{1-u}\right)$$

for u, to get

$$u = \frac{e^x}{1+e^x}.$$

So $X = F^{-1}(U)$ where

$$F(x) = \frac{e^x}{1+e^x}.$$

This F is a CDF (by the properties of a CDF, as discussed in class). So by Universality of the Uniform, $X \sim F$.

(c) By symmetry, $1-U$ has the same distribution as U, so by linearity,

$$E(X) = E(\ln U - \ln(1-U)) = E(\ln U) - E(\ln(1-U)) = 0.$$

Question 4.

Let X and Y be positive random variables, *not necessarily independent.* Assume that the various expected values below exist. Write the most appropriate of $\leq$, $\geq$, $=$, or ? in your answer booklet for each part (where "?" means that no relation holds in general). Write down the name of the rules, theorems or inequalities you used.

(a) $(E(XY))^2$ ____ $E(X^2)E(Y^2)$

(b) $P(|X+Y|>2)$ ____ $\frac{1}{10}E((X+Y)^4)$

(c) $E(\ln(X+3))$ ____ $\ln(E(X+3))$

(d) $E(X^2e^X)$ ____ $E(X^2)E(e^X)$

(e) $P(X+Y=2)$ ____ $P(X=1)P(Y=1)$

(f) $P(X \geq 1 \text{ or } Y \geq 1)$ ____ $P(\{X \geq 1\} \cup \{Y \geq 1\})$

Solution 4.

(a) $(E(XY))^2 \leq E(X^2)E(Y^2)$ (by Cauchy-Schwarz and than taking the square of both sides)

(b) $P(|X+Y|>2) \leq \frac{1}{10}E((X+Y)^4)$ (by Markov's Inequality $P(|X+Y|>2) \leq \frac{1}{16}E((X+Y)^4) \leq \frac{1}{10}E((X+Y)^4)$)

(c) $E(\ln(X+3)) \leq \ln(E(X+3))$ (by Jensen for concave functions)

(d) $E(X^2e^X) \geq E(X^2)E(e^X)$ (since X^2 and e^X have a positive covariance on the positive domain, as both increase in X)

(e) $P(X+Y=2)\,?\,P(X=1)P(Y=1)$
(What if X, Y are independent? What if $X \sim \text{Bern}(1/2)$ and $Y = 1 - X$?)

(f) $P(X \geq 1 \text{ or } Y \geq 1) = P(\{X \geq 1\} \cup \{Y \geq 1\})$ (by the definition of the union)

Question 5.

(a) A woman is pregnant with twin boys. Twins may be either identical or fraternal (non-identical). In general, 1/3 of twins born are identical. Obviously, identical twins must be of the same sex; fraternal twins may or may not be. Assume that identical twins are equally likely to be both boys or both girls, while for fraternal twins all possibilities are equally likely. Given the above information, what is the probability that the woman's twins are identical?

(b) A certain genetic characteristic is of interest. For a random person, this has a numerical value given by a $\mathcal{N}(0, \sigma^2)$ random variable. Let X_1 and X_2 be the values of the genetic characteristic for the twin boys from (a). If they are identical, then $X_1 = X_2$; if they are fraternal, then X_1 and X_2 have correlation ρ.
   (i) Find the value of $E(X_2^2)$ in terms of σ^2.

   (ii) Find $\text{Cov}(X_1, X_2)$ in terms of ρ, σ^2.

Solution 5.

(a) By Bayes' Rule and the law of total probability,

$$P(\text{identical} \mid BB) = \frac{P(BB \mid \text{identical})P(\text{identical})}{P(BB)} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{2}{3}} = \frac{1}{2}.$$

(b) (i) Since the means are zero, $E(X_2)^2 = 0$. Hence, $\text{Var}(X_2) = E(X_2^2) = \sigma^2$

   (ii) Since the means are 0,

$$\text{Cov}(X_1, X_2) = E(X_1 X_2) - (E(X_1))(E(X_2)) = E(X_1 X_2).$$

We find this by conditioning on whether the twins are identical or fraternal:

$$E(X_1 X_2) = E(X_1 X_2 \mid \text{identical}) \cdot \frac{1}{2} + E(X_1 X_2 \mid \text{fraternal}) \cdot \frac{1}{2}.$$

For identical twins, $E(X_1 X_2 \mid \text{identical}) = E(X_1^2) = \sigma^2$. For fraternal twins,

$E(X_1 X_2 \mid \text{fraternal}) = Cov(X_1, X_2 \mid \text{fraternal})$

$= Corr(X_1, X_2 \mid \text{fraternal}) \cdot \sqrt{Var(X_1 \mid \text{fraternal}) Var(X_2 \mid \text{fraternal})} = \rho\sigma^2.$

Thus:

$$E(X_1 X_2) = \sigma^2 \cdot \frac{1}{2} + \rho\sigma^2 \cdot \frac{1}{2} = \frac{\sigma^2}{2}(1 + \rho).$$

Therefore:

$$\text{Cov}(X_1, X_2) = \frac{\sigma^2}{2}(1 + \rho).$$

Question 6.

Cassie enters a casino with $X_0 = 1$ dollar and repeatedly plays the following game: with probability 1/3, the amount of money she has increases by a factor of 3; with probability 2/3, the amount of money she has decreases by a factor of 3. Let X_n be the amount of money she has after playing this game n times. For example, X_1 is 3 with probability 1/3 and is 3^{-1} with probability 2/3.

(a) Compute $E(X_1)$, $E(X_2|X_1 = \frac{1}{3})$ and $E(X_2|X_1 = 3)$.

(b) Compute $E(X_2)$ and, in general, $E(X_n)$.

(c) What happens to $E(X_n)$ as $n \to \infty$?

(d) Let Y_n be the number of times out of the first n games that Cassie triples her money. What happens to Y_n/n as $n \to \infty$? Which Theorem justifies your answer?

(e) Find an expression for X_n in terms of Y_n.

(f) What happens to X_n as $n \to \infty$?

Solution 6.

(a) Denote with W_i that Cassie wins in round i, that is, she triples her money. By the law of total expectation $E(X_1) = E(X_1|W_1)P(W_1) + E(X_1|W_1^c)P(W_1^c) = 3 \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = 1 + \frac{2}{9} = \frac{11}{9}$.
Similarly,

$$E(X_2|X_1 = \frac{1}{3}) = E(X_2|X_1 = \frac{1}{3}, W_2)P(W_2) + E(X_2|X_1 = \frac{1}{3}, W_2^c)P(W_2^c)$$
$$= 1 \cdot \frac{1}{3} + \frac{1}{9} \cdot \frac{2}{3} = \frac{1}{3} + \frac{2}{27} = \frac{11}{27},$$

and

$$E(X_2|X_1 = 3) = E(X_2|X_1 = 3, W_2)P(W_2) + E(X_2|X_1 = 3, W_2^c)P(W_2^c)$$
$$= 9 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = 3 + \frac{2}{3} = \frac{11}{3}.$$

(b) By the law of total expectation, and from the fact that the events $W_1 = \{X_1 = 3\}$ and $W_1^c = \{X_1 = \frac{1}{3}\}$ are equal

$$E(X_2) = E(X_2|X_1 = 3)P(X_1 = 3) + E(X_2|X_1 = \frac{1}{3})P(X_1 = \frac{1}{3})$$
$$= E(X_2|X_1 = 3)P(W_1) + E(X_2|X_1 = \frac{1}{3})P(W_1^c) = \frac{11}{3} \cdot \frac{1}{3} + \frac{11}{27} \cdot \frac{2}{3} = \frac{11}{9} + \frac{22}{81} = \frac{121}{81} = \left(\frac{11}{9}\right)^2$$

At this point you can realize that $E(X_n) = \left(\frac{11}{9}\right)^n$, or can argue as follows: Denote with G_i the factor of gain in round i, that are iid random variables with distribution $P(G_i = 3) = \frac{1}{3}$, $P(G_i = \frac{1}{3}) = \frac{2}{3}$ and $P(G_i = k) = 0$ for any $k \notin \{3, \frac{1}{3}\}$. At any round n, the amount of money she has is equal to her starting money ($X_0 = 1\$$) multiplied by the gain factors of all the preceding rounds, that is $X_n = X_0 \cdot G_1 \dots G_n$. As the G_i-s are iid, and X_0 is a constant $E(X_n) = 1 \cdot E(G_1) \dots E(G_n) = E(G_1)^n$. The expected gain factor is by the definition of the expectation is

$$E(G_1) = 3 \cdot P(G_1 = 3) + \frac{1}{3} \cdot P(G_1 = \frac{1}{3}) = 3 \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{11}{9},$$

hence $E(X_n) = (\frac{11}{9})^n$

(c) Since $E(X_n) = \left(\frac{11}{9}\right)^n$, if $n \to \infty$, $E(X_n) \to \infty$.

(d) Using the previous notation $Y_n = \mathbb{I}(G_1 = 3) + \dots + \mathbb{I}(G_n = 3)$, where $\mathbb{I}(G_i = 3)$ are iid Bernoulli random variables with parameter $p = \frac{1}{3}$, hence they have mean/expectation $E(\mathbb{I}(G_i = 3)) = \frac{1}{3}$. The law of large numbers says that the sample mean $\frac{\mathbb{I}(G_1=3)+\dots+\mathbb{I}(G_n=3)}{n}$ converges to the true mean $\frac{1}{3}$, with probability 1, as $n \to \infty$, that is, $P(\frac{Y_n}{n} \to \frac{1}{3}) = 1$.

(e) As argued in (b) $X_n = G_1 \dots G_n = 3^{\sum_{i=1}^n \mathbb{I}(G_i=3)} \cdot \left(\frac{1}{3}\right)^{n - \sum_{i=1}^n \mathbb{I}(G_i=3)} = 3^{Y_n} \cdot \left(\frac{1}{3}\right)^{n-Y_n}$.

(f) By part (e)

$$X_n = 3^{Y_n} \cdot \left(\frac{1}{3}\right)^{n-Y_n} = \left(3^{\frac{Y_n}{n}} \cdot \left(\frac{1}{3}\right)^{1-\frac{Y_n}{n}}\right)^n,$$

and by taking the logarithm of both sides

$$\log(X_n) = n \cdot \log\left(3^{\frac{Y_n}{n}} \cdot \left(\frac{1}{3}\right)^{1-\frac{Y_n}{n}}\right)$$

We showed in part (d) that as $n \to \infty$, $P(\frac{Y_n}{n} \to \frac{1}{3}) = 1$, thus

$$P\left(3^{\frac{Y_n}{n}} \cdot \left(\frac{1}{3}\right)^{1-\frac{Y_n}{n}} \to 3^{1/3} \cdot \left(\frac{1}{3}\right)^{2/3}\right) = 1.$$

Note that $3^{1/3} \cdot \left(\frac{1}{3}\right)^{2/3} = \left(\frac{1}{3}\right)^{1/3} < 1$, thus $\log\left(\left(\frac{1}{3}\right)^{1/3}\right) < 0$. Therefore, $\log(X_n) = n \cdot \log\left(3^{\frac{Y_n}{n}} \cdot \left(\frac{1}{3}\right)^{1-\frac{Y_n}{n}}\right)$ diverges to minus infinity with probability 1, implying that since $X_n = \exp(\log(X_n))$

$$P(X_n \to 0) = 1$$

as $n \to \infty$.